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ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

The Brauer-Grothendieck group

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Master project

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Introduction

The aim of this project is to introduce the Brauer group and the Brauer-Grothendieck group of a scheme. The Brauer group of a scheme, which is denoted by $\mathrm{Br}(X)$, is the generalization of the Brauer group of a field: we consider Azumaya algebras over a scheme instead of central simple algebras over the field. The cohomological Brauer group, or Brauer-Grothendieck group, is defined via the étale cohomology as $\mathrm{Br}'(X) = H^2(X_{\text{ét}}, \mathbb{G}_m)$. This group is an fundamental invariant associated to scheme and is used to define the *Manin obstruction* which gives useful information on the failure of the Hasse principle on a variety. In the next section, we present some information about this Manin obstruction. The reference is [Sko01].

In what follows, we use the following conventions: k is a field and Ω_k is its set of places (recall that a place is an equivalence classes of valuation of k). If $v \in \Omega_k$ is a place of k, we denote by k_v the completion of k with respect to v. We also denote $\prod_{v\in\Omega_k} k_v$ by k_{Ω_k}. If X is a scheme defined over k, we denote by $X(k)$ its set of rational points, endowed with the Zariski topology.

Diophantine equations, the Hasse principle and the Manin obstruction

Diophantine equations (indeterminate polynomial equations which admissible solutions are integers) are studied by mathematicians since the early development of mathematics. Since 1970 we know there is no algorithm which determine if a Diophantine equation with integer coefficients has a solution or not (see [Mat93]). However, the existence of solutions of such equations over $\mathbb{Q}$ is still an open problem. One obvious necessary condition is the existence of a solution over $\mathbb{R}$ and over $\mathbb{Q}_p$ for every $p \in \mathbb{P}$. The natural question is whether this is a sufficient condition. If it is the case, we say that the equation satisfies the Hasse principle. More precisely, we have the following.

Definition (Hasse principle)
Let k be a global field, let Ω_k be its set of places and let $\mathcal{C}$ be a class of algebraic varieties over k. We say that $\mathcal{C}$ satisfies the Hasse principle *if for every $V \in \mathcal{C}$ we have*

$$V(k_v) \neq \emptyset, \forall v \in \Omega_k \Rightarrow V(k) \neq \emptyset.$$

We say that a collection of equations satisfies the Hasse principle if the class of varieties defined by these equations satisfies the Hasse principle.

For example, if an homogeneous polynomial $f(x_1, \ldots, x_n)$ with coefficients in $\mathbb{Z}$ satisfies the Hasse principle, then it means that if the equation $f(x_1, \ldots, x_n) = 0$ has a solution in $\mathbb{R}$ and $\mathbb{Q}_p$ for every $p \in \mathbb{P}$, then $f(x_1, \ldots, x_n) = 0$ has a solution in $\mathbb{Z}$.

The *Hasse-Minkowski theorem* (see [Ser73, Chapter 4, §3, Theorem 8]) implies that the class of varieties defined by a quadratic form satisfies the Hasse principle. It is well know that there exists simple equations which do not satisfy the Hasse principle. For example, the equation $3x^3 + 4y^3 + 5z^3 = 0$ has solutions in every completion of $\mathbb{Q}$ but not in $\mathbb{Q}$ (see [Sel64]). A lot of counter examples of the Hasse principle were provided by mathematicians but these examples seemed somehow unrelated. In 1970, Yuri Manin presented a general obstruction, the Manin obstruction, which explained all the counter-examples to the Hasse principle which existed at that time[1]. We can prove for some classes of varieties that the Manin obstruction is the only obstruction to the Hasse principle. Moreover, in some cases (for example when the Picard group has good properties) one can easily compute the Manin obstruction.

The Manin obstruction

Let Y be a scheme and let X and Y' be two schemes over Y. We can define the following pairing:

$$\begin{aligned} \mathrm{Br}'(X) \times X(Y') &\longrightarrow \mathrm{Br}'(Y') \\ (A, s) &\longmapsto P(s) := s^*(A), \end{aligned}$$

where $\mathrm{Br}'(X)$ denotes the cohomological Brauer group of X (see Section 3.3). In particular, if X is a smooth variety over a field k, then for every extension K of k, we have the pairing

$$\begin{aligned} \mathrm{Br}'(X) \times X(K) &\longrightarrow \mathrm{Br}(K) \\ (A, s) &\longmapsto A(s) := s^*(A). \end{aligned}$$

Now, suppose that X is a smooth and geometrically integral variety over a number field k. Recall that we have the exact sequence

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \textstyle\sum_{v\in\Omega_k} \mathrm{Br}(k_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

where Ω_k is the set of places of k, the second arrow is the diagonal map and the third arrow is the sum of the local invariant maps $\mathrm{inv}_v : \mathrm{Br}(k_v) \hookrightarrow \mathbb{Q}/\mathbb{Z}$. These local invariant maps give rise to the *adelic Brauer-Manin pairing*[2]:

$$\begin{aligned} \mathrm{Br}'(X) \times X(\mathbb{A}_k) &\longrightarrow \mathbb{Q}/\mathbb{Z} \\ \big(A, (P_v)\big) &\longmapsto \sum_{v\in\Omega_k} \mathrm{inv}_v \big(A(P_v)\big), \end{aligned}$$

where $X(\mathbb{A}_k) = \prod_{v\in\Omega_k} X(k_v)$ is the set of adelic points. Note that if the variety X is such that $X(\mathbb{A}_k) \neq \emptyset$ and $X(k) = \emptyset$, then X is a counter-example to the Hasse principle.

Now, we define $X(\mathbb{A}_k)^{\mathrm{Br}'(X)}$ as the right kernel of this map, i.e. the set

$$X(\mathbb{A}_k)^{\mathrm{Br}'(X)} = \big\{(P_v)_v : \big(A, (P_v)_v\big) \longmapsto 0, \forall A \in \mathrm{Br}'(X)\big\}.$$

[1] Alexei Skorobogatov gave a counter-example to the Hasse principle which is not explained by the Manin obstruction (see [Sko99]). More recently, Bjorn Poonen showed the construction of a k-variety X (for a global field k of characteristic not 2) such that $X(k) = \emptyset$ but for which the emptiness cannot be explained by the Brauer-Manin (see [Poo10]).

[2] The fact that the sum is indeed finite is showed in [Sko01, 5.2].

It can be shown that the image of $X(k)$ under the inclusion $X(k) \hookrightarrow X(\mathbb{A}_k)$ is contained in $X(\mathbb{A}_k)^{\mathrm{Br}'(X)}$. Therefore, we have the following chain for every A in $\mathrm{Br}'(X)$:

$$X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}'(X)} \subset X(\mathbb{A}_k)^A \subset X(\mathbb{A}_k).$$

In particular, if $X(\mathbb{A}_k)^{\mathrm{Br}'(X)} = \emptyset$, there is no k-rational point.

Definition (Counter-example accounted for by the Manin obstruction)
If the variety X is such that $X(\mathbb{A}_k) \neq \emptyset$ and $X(k) = \emptyset$, then X is a counter-example to the Hasse principle. In this case, if $X(\mathbb{A}_k)^{\mathrm{Br}'(X)} = \emptyset$, we say that this counter-example is accounted for by the Manin obstruction.

In a similar way, using the unramified Brauer group $\mathrm{Br}_{\mathrm{ur}}(X)$[3] we can define the *Manin obstruction to the weak approximation*: we consider the

$$\begin{aligned} \mathrm{Br}_{\mathrm{ur}}(X) \times X(k_\Omega) &\longrightarrow \mathbb{Q}/\mathbb{Z} \\ \big(A, (P_v)\big) &\longmapsto \sum_{v \in \Omega_k} \mathrm{inv}_v \big(A(P_v)\big), \end{aligned}$$

and look at its right kernel $X(k_\Omega)^{\mathrm{Br}_{\mathrm{ur}}(X)}$. The set $X(k_\Omega)^{\mathrm{Br}_{\mathrm{ur}}(X)}$ allows us to give an explicit condition whether the Manin obstruction is the only obstruction to the weak approximation and to the Hasse principle.

Content of this project

This project consists of three parts. In the first one, we present some prerequisites such as group cohomology, profinite groups, Galois cohomology together with a few results about sheaves and sheaf cohomology. The second part is dedicated to the presentation of the Brauer group of a field; we give three equivalent definitions of the Brauer-Group and study some examples (finite fields, quasi-algebraically closed fields and local fields).
The lasts part of this project is dedicated to the presentation of the Brauer group of a scheme. We present the two definitions of this Brauer group.

[3]Recall that the unramified Brauer group of a field extension K/k is defined as $\mathrm{Br}_{\mathrm{ur}}(K/k) = \bigcap_R \mathrm{im}\big(\mathrm{Br}(R) \longrightarrow \mathrm{Br}(K)\big)$, where R runs through the set of discrete valuation rings such that $k \subset R \subset K$ and K is the field of fraction of R. The unramified Brauer group of our variety X is $\mathrm{Br}_{\mathrm{ur}}(X) := \mathrm{Br}_{\mathrm{ur}}(k(x)/k)$

Chapter 1

Prerequisites

1.1 Cohomology

1.1.1 Group cohomology

In this section, we recall briefly some definitions and results about groups cohomology. Let G be a group and let A be a left $\mathbb{Z}G$-module. We choose some projective resolution $P_\bullet$ of $\mathbb{Z}$, considered as a trivial $\mathbb{Z}G$-module, that is we have the following exact sequence of $\mathbb{Z}G$-modules:

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

Applying the left exact functor $\mathrm{Hom}_{\mathbb{Z}G}(-, A)$ gives rise to a complex of abelian groups

$$\mathrm{Hom}_{\mathbb{Z}G}(P_0, A) \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(P_1, A) \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(P_2, A) \longrightarrow \cdots.$$

Now, we can compute the cohomology groups of the complex $\mathrm{Hom}_{\mathbb{Z}G}(P_\bullet, A)$. Using standard theorems of homological algebra (see for example [Rot08, Comparison Theorem]), one can show that if $P'_\bullet$ is another projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$, then the cohomology groups of the two complexes $\mathrm{Hom}_{\mathbb{Z}G}(P_\bullet, A)$ and $\mathrm{Hom}_{\mathbb{Z}G}(P'_\bullet, A)$ are canonically isomorphic. This motivates the following definition:

Definition 1.1.1 (n-th cohomology group)
Let G, $P_\bullet$ and A be as before. The n-th cohomology group of G with coefficients in *A, denoted by $H^n(G, A)$, is the n-th cohomology group of the complex* $\mathrm{Hom}_{\mathbb{Z}G}(P_\bullet, A)$.

We have the following basic properties.

Proposition 1.1.2
Let G be a group and let A be a $\mathbb{Z}G$-module. Then, we have $H^0(G, A) = A^G$, the G-fixed point of A.

Proposition 1.1.3
Let G be a group. Then, $H^n(G, -)$ is a covariant functor from the category of $\mathbb{Z}G$-modules to the category of abelian groups.

Theorem 1.1.4 (Long exact sequence in cohomology)
Let G be a group and let $0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$ be a short exact sequence

of $\mathbb{Z}G$-modules. Then, there exists for each $n \in \mathbb{N}$ a connecting homomorphism $H^n(G,C) \longrightarrow H^{n+1}(G,A)$ such that the following sequence is exact:

$$0 \longrightarrow H^0(G,A) \xrightarrow{\varphi_*} H^0(G,B) \xrightarrow{\psi_*} H^0(G,C) \xrightarrow{\delta} H^1(G,A) \longrightarrow \cdots$$
$$\cdots \longrightarrow H^n(G,A) \xrightarrow{\varphi_*} H^n(G,B) \xrightarrow{\psi_*} H^n(G,C) \xrightarrow{\delta} H^{n+1}(G,A) \longrightarrow \cdots.$$

Proof. See [Rot08, Corollary 6.46]. □

Example 1.1.5 (Cohomology of finite cyclic groups)
Let $G = \langle g \rangle$ be a cyclic group of order n and let A be a $\mathbb{Z}G$-module. We want to compute the cohomology groups $H^m(G,A)$. For any $x \in \mathbb{Z}G$, we denote by $\mathrm{mult}_x : \mathbb{Z}G \longrightarrow \mathbb{Z}G$ the morphism of $\mathbb{Z}G$-modules which send some element a to $a \cdot x$. We also let $\lambda = \sum_{i=0}^{n-1} g^i \in \mathbb{Z}G$. Then, one can check that the following sequence is a projective (in fact free) $\mathbb{Z}G$-resolution of $\mathbb{Z}$:

$$\cdots \longrightarrow \mathbb{Z}G \xrightarrow{\mathrm{mult}_\lambda} \mathbb{Z}G \xrightarrow{\mathrm{mult}_{g-1}} \mathbb{Z}G \xrightarrow{\mathrm{mult}_\lambda} \mathbb{Z}G \xrightarrow{\mathrm{mult}_{g-1}} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

where $\varepsilon(g) = 1$ (remark that the kernel of ε is generated by $\{g^i - 1 : 1 \leq i \leq n-1\}$). Using the isomorphism $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) \cong A$ (as $\mathbb{Z}G$-modules), any morphism $\mathrm{mult}_x^* : \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A)$ becomes $\mathrm{mult}_x : A \longrightarrow A$. Therefore, applying the functor $\mathrm{Hom}_{\mathbb{Z}G}(-, A)$ to the projective resolution gives the complex

$$A \xrightarrow{\mathrm{mult}_{g-1}} A \xrightarrow{\mathrm{mult}_\lambda} A \xrightarrow{\mathrm{mult}_{g-1}} A \xrightarrow{\mathrm{mult}_\lambda} A \longrightarrow \cdots.$$

Finally, we have

$$H^m(C_n, A) = \begin{cases} A^{C_n} & \text{if } m = 0, \\ A^{C_n}/\mathrm{im}\,\mathrm{mult}_\lambda & \text{if } m \text{ is even}, m \geq 2, \\ \ker \mathrm{mult}_\lambda / \mathrm{im}\,\mathrm{mult}_{g-1} & \text{if } m \text{ is odd.} \end{cases}$$

Proposition 1.1.6
Let G be a finite group. Then, $H^m(G,\mathbb{Q}) = 0$ for every $m \geq 1$.

Proof. Let $n = |G|$. It is well known that $n \cdot H^m(G,\mathbb{Q}) = 0$. Now, the map $\mathbb{Q} \longrightarrow \mathbb{Q}$ which sends any $q \in \mathbb{Q}$ to $n \cdot q$ is an automorphism of $\mathbb{Q}$ which gives rise to an automorphism $H^m(G,\mathbb{Q}) \longrightarrow H^m(G,\mathbb{Q})$ which corresponds to the multiplication by n, as required. □

1.1.2 Profinite groups

Definition 1.1.7 (Profinite group)
Let G be a topological group. We say that G is profinite *if it is compact, Hausdorff and totally disconnected.*

Theorem 1.1.8
Let G be a topological group. Then, G is profinite if and only if there exists a fundamental system of open neighbourhood $\mathcal{I}$ of the neutral element consisting of normal subgroups of G such that $\bigcap_{N \in \mathcal{I}} N = \{1\}$ and $G \cong \varprojlim_{N \in \mathcal{I}} G/N$.

This characterization is interesting because of the following construction. Let k be a field and let K be a Galois extension of k, let $G = \mathrm{Gal}(K, k)$ be the Galois group of K over k. For each finite Galois extension $E \subset K$ of k, we denote by G'_E the group $\mathrm{Gal}(K, E)$, which is a normal subgroup of G. Now, if $F \subset K$ is another finite Galois extension of k, then so is the composite $E \vee F$. Moreover, we have $G'_E \cap G'_F = G'_{E \vee F}$. Therefore, the set $\{G'_E\}_E$, where E runs through the set of finite Galois extension contained in K, is a neighbourhood system consisting of normal subgroups. Thus, there exists a unique topology on G which is compatible with the group structure of G (see [Bou07]).

Definition 1.1.9 (Krull topology)
The topology defined above is called the Krull topology.

Theorem 1.1.10
Let k be a field, let K be a Galois extension of k and let $G = \mathrm{Gal}(K, k)$. Then, we have

$$G \cong \varprojlim_E \mathrm{Gal}(E, k),$$

where E runs through the set of finite Galois extension contained in K. In particular, G is a profinite group.

Examples 1.1.11 (i) If we let $p \in \mathbb{P}$, then $\mathrm{Gal}\left(\overline{\mathbb{F}_p}, \mathbb{F}_p\right) \cong \hat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$. More generally, if F is a finite field, then $\mathrm{Gal}\left(\overline{F}, F\right) \cong \hat{\mathbb{Z}}$ (this is because the set of finite Galois extensions of F is cofinal in the set of finite Galois extensions of $\mathbb{F}_p$, where p is the characteristic of F).

(ii) Let p be an odd prime. For $m \in \mathbb{N}$, we denote by ζ_m a primitive m-th root of unity in $\overline{\mathbb{Q}}$ and $F_n = \mathbb{Q}[\zeta_{p^n}]$. Now, if we let $F = \bigcup_{n \geq 0} F_n$, then we have

$$\mathrm{Gal}\left(F, \mathbb{Q}[\zeta_p]\right) \cong \mathbb{Z}_p.$$

1.1.3 Cohomology of profinite groups

Proposition 1.1.12
Let G be a profinite group and let A be a G-module. The following are equivalent:

(i) The maps $G \times A \longrightarrow A$, which send (g, a) to $g \cdot a$ is continuous (A is endowed with the discrete topology).

(ii) For each $a \in A$, the stabilizer G_a is an open subgroup of G.

(iii) $\bigcup_{H \leq_o G} A^H = A$.

Definition 1.1.13 (Continuous G-module)
Let G be a profinite group and let A be a G-module. We say that A is a continuous G-module *(or that the* action is continuous*) if one of the condition of the previous definition is satisfied.*

Examples 1.1.14 (i) If G is finite, then every G-module is a continuous G-module.

(ii) Let K/k be a Galois field extension and $G = \mathrm{Gal}(K, k)$. Then, the action of G on K^* is continuous. To see this, let a be an element of K^*. Then $\sigma \in G$ is in G_a if and only if $\sigma \in \mathrm{Gal}(K, k[a])$. Now, if $f = \min(a, k)$ is the minimal polynomial of a over k, then $K[f]$, the splitting field of f, is a finite

Galois extension of k, which implies that $\mathrm{Gal}(K, K[f])$ is open in G. Since $\mathrm{Gal}(K, K[f]) \subset \mathrm{Gal}(K, k[a]) = G_a$, then G_a is open in G (in a topological group, each subgroup containing an open subset is open).

Let G be a profinite group and let A be a continuous G-module. Let $n \in \mathbb{N}_0$. If N', N are two open normal subgroups of G such that $N' \subset N$, we have an morphism of groups

$$H^n\big(G/_N, A^N\big) \longrightarrow H^n\big(G/_{N'}, A^{N'}\big).$$

These morphisms give rise to a direct system of groups $H^n\left(G/_N, A^N\right)$, where N runs through the set of open normal subgroups of G. This motivates the following definition:

Definition 1.1.15 (Cohomology groups of a profinite group)
Let G be a profinite group and let A be a G-module. The n-th cohomology group of G with coefficients in A *is* $H^n(G, A) = \varinjlim_{N \trianglelefteq_o G} H^n\left(G/_N, A^N\right)$.

Remark 1.1.16
Instead of using the limit, one could have defined the cohomology groups in the "usual way": we let $C^n(G, A)$ be the set of all *continuous* (recall that A is endowed with the discrete topology) maps from G^n to A and we consider the coboundaries $C^n(G, A) \longrightarrow C^{n+1}(G, A)$ defined as

$$\begin{aligned}(df)(g_1, \dots, g_{n+1}) = g_1 \cdot f\big(g_2, \dots, g_{n+1}\big) \\ + \sum_{i=1}^{n}(-1)^i f\big(g_1, \dots, g_{i-1}, g_i \cdot g_{i+1}, g_{i+2}, \dots, g_{n+1}\big) \\ + (-1)^{n+1} f\big(g_1, \dots, g_n\big).\end{aligned}$$

And then, consider the cohomology groups of the complex $\big(C^n(G, A), d\big)$. One can easily show that the two definitions are equivalent. However, the definition with the limit is very powerful because one can use results from the finite case, as shown in Proposition 1.1.18.

Remark 1.1.17
In particular, if $K|k$ is a Galois extension (not necessarily finite) and $G = \mathrm{Gal}(K, k)$, then we have

$$H^2(G, A) = \varinjlim_{F} H^2\Big(\mathrm{Gal}(F, k), A^{\mathrm{Gal}(K,F)}\Big),$$

where F runs through the set of finite Galois extension F of k contained in K.

Proposition 1.1.18
Let A be a continuous G-module and let $n \geq 1$. Then, the group $H^n(G, A)$ is a torsion group.

Proof. When G is finite, this is a well-known result. To conclude, we remark that a direct limit of torsion group is again torsion. □

1.1.4 Non-abelian group cohomology

When we defined the cohomology groups $H^n(G, A)$, we considered a $\mathbb{Z}G$-module A. Although it is not possible to define all the $H^n(G, A)$ if A is not commutative, it is possible to define a group $H^0(G, A)$ and a pointed set $H^1(G, A)$. The reference for this section is [Ser94, Chapter I, § 5]. In this section, G denotes a profinite group.

Definition 1.1.19 (G-set)
A G-set is a discrete topological space in which G acts continuously (see Proposition 1.1.12). If A is a G-set, then we denote $g \cdot a$ by ${}^g a$, for all $g \in G$ and $a \in A$.

Definition 1.1.20 (G-group)
Let A be a G-set. We say that A is a G-group if the action of G is compatible with the group structure of A, that is if ${}^g(ab) = {}^g a \cdot {}^g b$, for every $g \in G$ and every $a, b \in A$.

From now, A denotes a G-set.

Definition 1.1.21 (Cocycle, set of cocycles)
A 1-cocycle, or cocycle, of G with values in A, is a continues map a from G to A, $\sigma \longmapsto a_\sigma$ such that

$$a_{\sigma\tau} = a_\sigma \cdot {}^\sigma a_\tau, \quad \forall \sigma, \tau \in G.$$

We denote by $Z(G, A)$ the set of all such 1-cocycles.

We define an equivalence relation as follows: two cocycles a and a' are said to be *equivalent*, or *cohomologous*, if there exists $b \in A$ such that

$$a'_\sigma = b^{-1} \cdot a_\sigma \cdot {}^\sigma b, \quad \forall \sigma \in G.$$

The quotient of $Z(G, A)$ by this equivalence relation is denoted by $H^1(G, A)$ we also let $H^0(G, A) = A^G$, the G-fixed points of A.

Remarks 1.1.22 (i) The $H^1(G, A)$ may fail to be a group. However, it is a pointed set. The base point is given by the class of the trivial cocycle.

(ii) If A is commutative, then A is a G-module. Then, a 1-cocycle is a derivation and two cocycles are cohomologous if they differ from a principal derivation. Hence, $H^0(G, A)$ and $H^1(G, A)$ are the same as the two first cohomology groups if A is commutative.

The next two propositions are easy to prove and well-known results.

Proposition 1.1.23
The pointed sets $H^1\left(G/_U, A^U\right)$, where U runs through the set of open normal subgroups of G, is a direct system. Moreover, we have

$$H^1(G, A) \cong \varinjlim_{U \trianglelefteq G} H^1\left(G/_U, A^U\right).$$

Proposition 1.1.24
The set $H^1(G, A)$ is functorial in A. Moreover, if

$$1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 1$$

is an exact sequence of G-groups, then we have an exact sequence of G-sets

$$1 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \longrightarrow H^1(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, C).$$

Proposition 1.1.25
Let $1 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 1$ be an exact sequence of G-groups such that A is contained in the center of B. Then, we have an exact sequence of G-sets

$$1 \to A^G \to B^G \to C^G \to H^1(G, A) \to H^1(G, B) \xrightarrow{\psi_*} H^1(G, C) \xrightarrow{\delta} H^2(G, A).$$

Proof. We only have to construct δ and to show the exactness at $H^1(G,C)$. Let $c : G \longrightarrow C$ be a 1-cocycle. For each $\sigma \in G$, choose a preimage $b_\sigma \in B$ of c_σ under ψ. Since c is a cocycle, $b_\sigma \cdot {}^\sigma b_\tau \cdot {b_{\sigma\tau}}^{-1}$ is in the kernel of ψ for each $\sigma, \tau \in G$. Therefore, there exists a unique $a_{\sigma\tau} \in A$ such that $a_{\sigma\tau} = b_\sigma \cdot {}^\sigma b_\tau \cdot {b_{\sigma\tau}}^{-1}$. Hence, we can define a map

$$a = \delta(c) : G \times G \longrightarrow A, \quad (\sigma, \tau) \longmapsto a_{\sigma,\tau}.$$

To show that a is a 2-cocycle, we have to show that

$${}^\sigma a_{\tau,\eta} \cdot {a_{\sigma\tau,\eta}}^{-1} \cdot a_{\sigma,\tau\eta} \cdot {a_{\sigma,\tau}}^{-1} = 1.$$

Using the definition of a and the fact that A is abelian gives the equality. Similarly, one can show that δ passes to the quotient to a map $\delta : H^1(G,C) \longrightarrow H^2(G,A)$ and that its definition does not depend on the preimage b_σ of c_σ.
Now, if $c \in \operatorname{im}\psi_*$, then the application $b : G \longrightarrow B$ is a 1-cocycle, which implies that $a_{\sigma,\tau} = 1$ for each $\sigma, \tau \in G$. Finally, suppose that $c \in \ker\delta$ which implies the existence of a map $g : G \longrightarrow A$ such that

$$b_\sigma \cdot {}^\sigma b_\tau \cdot {b_{\sigma\tau}}^{-1} = {}^\sigma g(\tau) \cdot g(\sigma\tau)^{-1} \cdot g(\sigma).$$

Now, define $\tilde{b} : G \longrightarrow B$ as

$$\tilde{b}_\sigma = b_\sigma \cdot g(\sigma)^{-1}.$$

It is easy to see that $\tilde{b}$ is a 1-cocycle and that $\psi_*(\tilde{b}) = b$. Therefore, we have $\operatorname{im}\psi_* = \ker\delta$, as required. □

Theorem 1.1.26 (Hilbert's theorem 90)
Let K/k be a Galois extension. Then, for every $n \in \mathbb{N}$, we have

$$H^1\big(\operatorname{Gal}(K,k), \operatorname{GL}_n(K)\big) \cong \{1\}.$$

Proof. See [Gug10, Theorem 2.5.12]. □

Using the short exact sequence $1 \longrightarrow \operatorname{SL}_n(K) \longrightarrow \operatorname{GL}_n(K) \xrightarrow{\det} K^* \longrightarrow 1$, we can prove the following.

Corollary 1.1.27
Let K/k be a Galois extension. Then, for every $n \in \mathbb{N}$, we have

$$H^1\big(\operatorname{Gal}(K,k), \operatorname{SL}_n(K)\big) \cong \{1\}.$$

1.1.5 Twisted forms and Galois descent

The descent problem could be summarized as follows. Suppose we are given a Galois extension K/k, a collection of objects defined over k (for example vector spaces, tensors, schemes[1], algebras, ...) and a way to extend the scalar from k to K. If two objects defined over k are isomorphic over K, are they also isomorphic over k? The material presented here can be found in [Ser94] and [Ser95].
In this section, we briefly present the case of vector spaces and tensors. Let K/k be a Galois extension. Unless stated otherwise, a vector space is a finite dimension vector space over k.

[1] See section 1.3.3.3 for twisted forms of schemes.

Definition 1.1.28 (Tensor)
Let V denotes a finite dimensional k-vector space. Let $(p,q) \in \mathbb{N}_0 \times \mathbb{N}_0$. A tensor of type (p,q) *is an element Φ of $V^{\otimes p} \otimes_k \left(V^*\right)^{\otimes q}$. In what follows, we shall also use the word tensor for the pair (V,Φ).*

Proposition 1.1.29
Let V be a finite dimensional k-vector space. We have the following isomorphism

$$V^{\otimes p} \otimes_k \left(V^*\right)^{\otimes q} \cong \operatorname{Hom}_k\left(V^{\otimes q}, V^{\otimes p}\right).$$

Example 1.1.30
If $q = 2$, $p = 1$ and Φ satisfies an associativity condition, then (V,Φ) is just a k-algebra.

Notation 1.1.31
Let (V,Φ) and (W,Ψ) be two tensors of type (p,q) and let $f : V \longrightarrow W$ be a linear map. Then, if we consider Φ and Ψ as multi-linear maps, we can define the maps $f \cdot \Phi$ and $\Psi \cdot f$ by post-composition and composition by f in each factor of V.

Definition 1.1.32 (Morphism of tensors)
Let (V,Φ) and (W,Ψ) be two tensors of type (p,q). A morphism *from (V,Φ) to (W,Ψ) is a k-linear map $f : V \longrightarrow W$ such that $f \cdot \Phi = \Psi \cdot f$.*

Remark 1.1.33
If $f : V \longrightarrow W$ is an isomorphism, then we have a canonical map $f^{*-1} : V^* \longrightarrow W^*$. Then, the condition $f \cdot \Phi = \Psi \cdot f$ is equivalent to ask that

$$f^{\otimes p} \otimes f^{*-1^{\otimes q}}(\Phi) = \Psi,$$

where Φ is viewed as an element of $\operatorname{Hom}_k\left(V^{\otimes q}, V^{\otimes p}\right)$ and Ψ as an element of $\operatorname{Hom}_k\left(W^{\otimes q}, W^{\otimes p}\right)$.

Notation 1.1.34
Let (V,Φ) be a tensor. We denote by V_K the K-vector space $V \otimes_k K$ and by Φ_K the tensor $\Phi \otimes \mathrm{id}_K$. Moreover, we denote by $\operatorname{Aut}_K(\Phi)$ the set of all automorphisms of (V_K, Φ_K).

For the rest of this section, we fix a finite dimensional k-vector space and a tensor (V,Φ) of type (p,q).

Definition 1.1.35 (Twisted form)
*Let (W,Ψ) be a tensor of type (p,q). We say that (W,Ψ) is a $K|k$-*twisted form *of (V,Φ) if $(V_K,\Phi_K) \cong (W_K,\Psi_K)$. We denote by $\mathrm{TF}_K(V,\Phi)$, or just $\mathrm{TF}_K(\Phi)$, the set of all k-isomorphism classes of $K|k$-twisted forms of (V,Φ). This is a pointed set with base point the class of (V,Φ).*

1.1.5.1 Descent and cohomology

As before, we consider a Galois extension K/k with Galois group G, a finite dimensional k-vector space V and a tensor Φ on V. We want to show that there exists a bijection of pointed sets between $\mathrm{TF}_K(V,\Phi)$ and $H^1\big(G, \operatorname{Aut}_K(\Phi)\big)$. We consider on $\operatorname{Aut}_K(\Phi)$ the following G action:

$$\begin{aligned} G \times \operatorname{Aut}_K(\Phi) &\longrightarrow \operatorname{Aut}_K(\Phi) \\ (\sigma, f) &\longmapsto {}^{\sigma}f = (\mathrm{id}_V \otimes \sigma) \circ f \circ (\mathrm{id}_V \otimes \sigma^{-1}). \end{aligned}$$

We remark the following:

(i) The action is compatible with the group structure on $\mathrm{Aut}_K(\Phi)$.

(ii) The action is continuous. Indeed, if the automorphism f is represented by the matrix (a_{ij}), then ${}^{\sigma}f$ is represented by the matrix $\big(\sigma(a_{ij})\big)$. Proceeding as in Examples 1.1.14 (and taking some compositum fields) shows that the action is continuous.

Hence, is makes sense to consider the set $H^1\big(G, \mathrm{Aut}_K(\Phi)\big)$.

Theorem 1.1.36
Let K/k be a Galois extension K/k with Galois group G, a finite dimensional k-vector space V and a tensor Φ on V. There exists a bijection of pointed sets between $\mathrm{TF}_K(V,\Phi)$ and $H^1\big(G, \mathrm{Aut}_K(\Phi)\big)$.

Proof. First, we want to define a map $\theta : \mathrm{TF}_K(V,\Phi) \longrightarrow H^1\big(G, \mathrm{Aut}_K(\Phi)\big)$. Let (W,Ψ) be a $K|k$-twisted form of Φ. By hypothesis, there exists a K-isomorphism $f : V_K \longrightarrow W_K$ which send Φ to Ψ. We associate to (W,Ψ) the following 1-cocycle:

$$\begin{aligned} a : G &\longrightarrow \mathrm{Aut}_K(\Phi) \\ \sigma &\longmapsto a_\sigma = f^{-1} \circ {}^{\sigma}f. \end{aligned}$$

It is easy to see that the class of the cocycle a in $H^1\big(G, \mathrm{Aut}_K(\Phi)\big)$ does not depend on the choice of the isomorphism f. Indeed, if $g : V_K \longrightarrow W_K$ is another K-isomorphism which maps Φ to Ψ and if we denote the corresponding cocycle by a', we have $a' \sim a$, via $b = g \circ f^{-1}$.

Injectivity of θ Suppose that (W,Ψ) and (W',Ψ') are two $K|k$-twisted forms which are mapped to the same element. If we denote by f and f' the K-isomorphisms $V_K \longrightarrow W_K$ and $V_K \longrightarrow W'_K$, then there exists $h \in \mathrm{Aut}_K(\Phi)$ such that

$$f'^{-1} \circ \sigma \circ \ f' = h^{-1} \circ f^{-1} \circ \sigma \circ f \circ h, \quad \forall \sigma \in G.$$

Hence, the element $f' \circ h^{-1} \circ f^{-1}$ is invariant under the action of G, which means that it comes from a k-isomorphism from W to W'. Hence, θ is injective.

Surjectivity of θ Let $a : G \longrightarrow \mathrm{Aut}_K(\Phi)$ be a 1-cocycle. Choosing a base of V_K allows us to view a as a cocycle with values in $\mathrm{GL}_n(K)$, for some n. Since $H^1\big(G, \mathrm{GL}_n(K)\big)$ is trivial (see Theorem 1.1.26), there exists $f \in \mathrm{Aut}_K(\Phi)$ such that $a_\sigma = f^{-1} \circ {}^{\sigma}f$, for every $\sigma \in G$. Let $\Psi = f(\Phi)$. We want to show that Ψ comes from V (hence it will be clear that (V,Ψ) is a preimage of a). To see this, let $\sigma \in G$ and compute

$$\sigma \cdot \Psi = \sigma \cdot f(\Phi) = \big({}^{\sigma}f\big)(\sigma \cdot \Phi) = \big({}^{\sigma}f\big)(\Phi) = \big(f \circ a_\sigma\big)(\Phi) = f(\Phi) = \Psi,$$

as required.

□

As mentioned before, an algebra can be viewed as a $(1,2)$-tensor which satisfies an associativity condition. Therefore, the previous theorem will help us to show that the Brauer group is isomorphic to some H^1.

1.2 Reduced norm and unramified extensions

1.2.1 Reduced norm

Let A be a finite dimensional central simple algebra over a field k. We will see (Theorem 2.1.5) that there exists a finite extension K of k and an integer n (in fact, n is the degree of A, see Definition 2.1.8) such that there exists a isomorphism $f : A \otimes_k K \longrightarrow \mathrm{M}_n(K)$. For an element $a \in A$, we can consider

$$\mathrm{Nrd}_{f,K}(a) = \det\big(f(a \otimes 1) - x \cdot I_n\big).$$

One can show that $\mathrm{Nrd}_{f,K}(a)$ does not depend on the choice of the isomorphism f and the splitting field K (see [Bou11, §12, n° 3]). Hence, we have the following definition.

Definition 1.2.1 (Reduced norm map)
Let A be a finite dimensional central simple algebra over a field k and chose an isomorphism $f : A \otimes_k K \longrightarrow \mathrm{M}_n(K)$. The reduced norm *of A is the map*

$$\begin{aligned}\mathrm{Nrd} : A &\longrightarrow K\\ a &\longmapsto \det\big(f(a \otimes 1) - x \cdot I_n\big).\end{aligned}$$

Proposition 1.2.2
Let A be as above. Then, the following hold:

(*i*) $\operatorname{im}\mathrm{Nrd} \subset k$;

(*ii*) Nrd *is multiplicative;*

(*iii*) *if $a \in k$, then $\mathrm{Nrd}(a) = a^n$, where $n = \sqrt{\dim_k A}$;*

(*iv*) $\mathrm{Nrd}(a) \neq 0$ *if and only if $a \in A^*$.*

Proof. (i) See [Bou11, §12, n° 3].

(ii) Clear.

(iii) $\mathrm{Nrd}(a) = \det\big(f(a \otimes 1)\big) = a^n \det\big(f(1 \otimes 1)\big)$.

(iv) If $a \in A^*$, it is clear that $\mathrm{Nrd}(a) \neq 0$ since Nrd is multiplicative. Now, suppose that $\mathrm{Nrd}(a) \neq 0$, which means that $f(a \otimes 1)$ and $a \otimes 1$ are invertible. Since A is artinian, we get that either a is invertible or a is a zero divisor. If a is a zero divisor, then $a \otimes 1$ cannot be invertible, contradiction. Hence, a is invertible, as required.

□

1.2.2 A few results about unramified extensions

We briefly present a few results about unramified extensions. The proofs and the details can be found in [Ser95]. We consider a complete discrete valuation ring (A, v) with residue field κ.

Theorem 1.2.3
For every finite separable extension κ' of κ, there exists an unramified finite extension K' of K such that the residual extension is isomorphic to κ'/κ. Moreover, K' is unique up to isomorphism and K'/K is Galois if and only if κ'/κ is Galois.

Definition 1.2.4 (Maximal unramified extension of a field)
Let $\mathcal{K}$ be the set of all finite and unramified extensions K' of K obtained by the previous theorem when κ' runs through the set of all finite and separable extension κ' of κ. We define the maximal unramified extension of K, *denoted by K_{ur} as follows:*

$$K_{\mathrm{ur}} = \varinjlim_{K' \in \mathcal{K}} K'.$$

Theorem 1.2.5
The maximal unramified extensions K_{ur} of K satisfies the following properties:

(*i*) *K_{ur}/K is Galois;*

(*ii*) *the residue field of K_{ur} is K_{s};*

(*iii*) $\mathrm{Gal}(K_{\mathrm{ur}}, K) = \mathrm{Gal}(\kappa_{\mathrm{s}}, \kappa)$.

Corollary 1.2.6
Let K''/K be a finite extension, with residue field κ''/κ. The subextensions K'/K of K''/K which are unramified over K are in one-to-one correspondence with separable subextensions κ'/κ of κ''/κ.

1.3 Sites and sheaf cohomology

1.3.1 A few things about sheaf cohomology

In this section, all schemes are assumed to be locally noetherian and X denotes such a scheme. Unless specified otherwise, a sheaf denotes a sheaf of abelian groups. Most of the material presented here can be found in [Mil80].

Definition 1.3.1 (Étale morphism)
Let $f : X \longrightarrow Y$ be a morphism locally of finite type between two schemes. Then, f is called étale *if it is flat and unramified.*

Notation 1.3.2
We denote by X_E the small E-site on X (see [Gug12]). For example, E can be:

$E = \mathbf{\acute{e}t}$ the class of étale morphisms;

$E = \mathbf{zar}$ the class of open immersions;

$E = \mathbf{fl}$ the class of flat morphisms.

Notation 1.3.3
We denote by $P(X)$ (respectively $S(X)$) the abelian categories of presheaves (respectively sheaves) on X_E with value in $\mathbf{Ab}$.

Theorem 1.3.4
The category $S(X)$ has enough injectives.

Proof. See [Mil80, Proposition III.1.1]. □

With this result, we can consider the right derived functor of any left exact functor from $S(X)$ to any abelian category (see [Gug11]).

Definition 1.3.5 (Cohomology groups)
Let $n \in \mathbb{N}_0$. It is well-known that the global section functor $\Gamma(X_E, -) : S(X) \longrightarrow \mathbf{Ab}$ is left exact. Its right derived functors are written

$$H^n(X, -) = H^n(X_E, -) := R^n\Gamma(X, -).$$

*If $\mathscr{F}$ is a sheaf on X_E, the group $H^n(X_E, \mathscr{F})$ is the n-*th cohomology group of X_E with values in $\mathscr{F}$.

Definition 1.3.6 (Ext groups)
Let $\mathscr{F}$ be a sheaf on X. The functor $\mathrm{Hom}_{X_E}(\mathscr{F}, -)$ is left exact and thus gives rise to its right derived functors $\mathrm{Ext}^n_{X_E}(\mathscr{F}, -)$.

Remark 1.3.7
Let $\mathbb{Z}_{X_E}$ be the constant sheaf defined by $\mathbb{Z}$ on X_E. Since we have a natural isomorphism $\mathscr{F}(X) = \mathrm{Hom}_{X_E}(\mathbb{Z}_{X_E}, \mathscr{F})$, then we have $\mathrm{Ext}^n_{X_E}(X, \mathscr{F}) = H^n(X_E, \mathscr{F})$ (see [Rot08, Corollary 6.49]).

1.3.2 The étale site on $\mathrm{Spec}\, k$ and étale cohomology of $\mathrm{Spec}\, k$

Let k be a field, let $X = \mathrm{Spec}\, k$ and let $G = \mathrm{Gal}(k_{\mathrm{s}}, k)$. An object $U \longrightarrow X$ of $X_{\text{ét}}$ can be written as a disjoint union of spectra of finite separable field extension of k, which means that U corresponds to a finite dimensional étale k-algebra A, that is $A \cong \bigoplus_i k_i$, where each k_i is a finite separable extension k_i of k (see [Mil80, Proposition I.3.2]). Now, we would like to identify the category of sheaves of abelian groups on $X_{\text{ét}}$. Let $\mathscr{F}$ be a presheaf of abelian groups on $X_{\text{ét}}$. For any finite separable extension k' of k, we write $\mathscr{F}(k')$ for $\mathscr{F}(\mathrm{Spec}\, k')$. If k'/k is Galois, then $\mathrm{Gal}(k', k)$ acts on $\mathscr{F}(k')$. It can be shown (cf [Mil80, Proposition II.1.4]) that if $\mathscr{F}$ sends disjoint union of schemes to direct product of abelian groups, then the sequence

$$\mathscr{F}(k') \longrightarrow \mathscr{F}(K) \rightrightarrows \mathscr{F}(K \otimes_{k'} K),$$

with k' finite over k and K finite Galois over k', is exact if and only if the equality $\mathscr{F}(k') = \mathscr{F}(K)^{\mathrm{Gal}(K,k')}$ holds.

Proposition 1.3.8
As above, let k be a field, let $X = \mathrm{Spec}\, k$ and let $G = \mathrm{Gal}(k_{\mathrm{s}}, k)$. Then, we have an equivalence of categories between the category of continuous G-modules (see Definition 1.1.13) and the category of sheaves on $X_{\text{ét}}$.

Proof. Let M be a continuous G-module. We define the following presheaf $\mathscr{F}_M$ on $X_{\text{ét}}$: an étale k-algebra A is mapped to

$$\mathscr{F}_M(A) = \mathrm{Hom}_G\left(\mathrm{Hom}_{k-\text{algebras}}(A, k_{\mathrm{s}})\right).$$

Note that this is equivalent to send an étale X-scheme U to

$$\mathscr{F}_M(U) = \mathrm{Hom}_G\left(\mathrm{Hom}_X(\overline{x}, U)\right),$$

where $\overline{x}$ is a geometric point of X. Now, the presheaf $\mathscr{F}_M$ satisfies:

$$\begin{aligned} \mathscr{F}_M : X_{\text{ét}} &\longrightarrow \mathbf{Ab} \\ \bigoplus_i k_i &\longmapsto \bigoplus_i M^{\mathrm{Gal}(k_{\mathrm{s}},k_i)}. \end{aligned}$$

We want to show that $\mathscr{F}_M$ is actually a sheaf. Thanks to [Mil80, Proposition II.1.5], it is sufficient to check that the following sequence

$$\mathscr{F}_M(U) \longrightarrow \prod_i \mathscr{F}_M(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}_M(U_i \times_U U_j)$$

is exact for:

(i) Zariski coverings;

(ii) étale coverings $U' \longrightarrow U$ with both U' and U affine.

The first point follows from the definition and properties of $\mathscr{F}_M$. For the second, we can suppose that U' and U correspond to a tower of separable extensions $k \subset F \subset F'$. Let L' be a finite Galois extension of F containing F'. We know (see above) that the sequence

$$\mathscr{F}_M(F) \longrightarrow \mathscr{F}_M(L') \rightrightarrows \mathscr{F}_M(L' \otimes_F L'),$$

is exact. Therefore, by considering the diagram

$$\begin{array}{ccccc} \mathscr{F}_M(F) & \longrightarrow & \mathscr{F}_M(F') & \rightrightarrows & \mathscr{F}_M(F' \otimes_F F') \\ \| & & \big\downarrow & & \big\downarrow \\ \mathscr{F}_M(F) & \longrightarrow & \mathscr{F}_M(L') & \rightrightarrows & \mathscr{F}_M(L' \otimes_F L'), \end{array}$$

one shows that the top row is also exact, as required.
If $\varphi : M \longrightarrow M'$ is a morphism of continuous G-modules, it induces a morphism of sheaves $\alpha_\varphi : \mathscr{F}_M \longrightarrow \mathscr{F}_{M'}$.
Now, let $\mathscr{F}$ be a sheaf on $X_{\text{ét}}$. We define

$$M_{\mathscr{F}} = \varinjlim_{k'} \mathscr{F}(k'),$$

where k' runs through the set $\mathcal{S}$ of finite (separable) extensions of k contained in k_{s}. Since the set of finite Galois extensions of k contained in k_{s} is cofinal in $\mathcal{S}$, we can suppose that each k' is Galois over k. In this case, the action of $\text{Gal}(k', k)$ on k' gives rise to an action of G on k' which itself turn $\mathscr{F}(K')$ into a G-module. Hence, $M_{\mathscr{F}}$ is a G-module and it is easy to check that it is a continuous G-module. If $\alpha : \mathscr{F} \longrightarrow \mathscr{F}'$ is a natural transformation between two sheaves on $X_{\text{ét}}$, then α induces a map of directed systems $\{\mathscr{F}(k')\} \longrightarrow \{\mathscr{F}'(k')\}$ which gives rise to a homomorphism of groups $M_{\mathscr{F}} \longrightarrow M_{\mathscr{F}'}$. Moreover, this homomorphism is a morphism of G-groups. Finally, we have

$$M_{\mathscr{F}_M} = \varinjlim_{k'} \mathscr{F}_M(k') = \varinjlim_{k'} M^{\text{Gal}(k_{\text{s}}, k')} = \bigcup_{H \leq_O G} M^H = M.$$

Hence, the functor which send $\mathscr{F}$ to $M_{\mathscr{F}}$ is (essentially) surjective and we have an equivalence of categories, as required. □

Corollary 1.3.9
Let $\mathscr{F}$ be a sheaf of abelian groups on $(\operatorname{Spec} k)_{\text{ét}}$. For every $n \in \mathbb{N}_0$, we have

$$H^n((\operatorname{Spec} k)_{\text{ét}}, \mathscr{F}) = H^n\big(\text{Gal}(k_{\text{s}}, k), \varinjlim_{k'} \mathscr{F}(k')\big),$$

where the limit is taken on the finite Galois extensions k' of k contained in k_{s}.

1.3.3 Čech cohomology

As in the "standard" sheaf cohomology, we have a more explicit way to compute cohomology: the Čech cohomology. We fix a scheme X and some E-covering $\mathcal{U} = \big\{ U_i \longrightarrow X \big\}_{i \in I}$ of X. For any element $(i_0, \dots, i_p)$ in I^{p+1} we denote $U_{i_0} \times_X \dots \times_X U_{i_p}$ by $U_{i_0,\dots,i_p}$. Let $\mathscr{F}$ be a presheaf on X_E. The projections $U_{i_0,\dots,i_p} \longrightarrow U_{i_0,\dots,\widehat{i_j},\dots,i_p}$ induce restriction maps

$$\mathrm{res}_j : \mathscr{F}\big(U_{i_0,\dots,\widehat{i_j},\dots,i_p}\big) \longrightarrow \mathscr{F}\big(U_{i_0,\dots,i_p}\big).$$

For each $p \in \mathbb{N}_0$, we define

$$C^p(\mathcal{U}, \mathscr{F}) = \prod_{(i_0,\dots,i_p) \in I^{p+1}} \mathscr{F}(U_{i_0,\dots,i_p})$$

and

$$d^p : C^p(\mathcal{U}, \mathscr{F}) \longrightarrow C^{p+1}(\mathcal{U}, \mathscr{F})$$
$$s \longmapsto d^p s, \quad (d^p s)_{i_0,\dots,i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \cdot \mathrm{res}_j(s_{i_0,\dots,\widehat{i_j},\dots,i_{p+1}}).$$

One can check that $\big(C^p(\mathcal{U}, \mathscr{F}), d^p\big)$ is a cochain complex and thus compute its cohomology.

Definition 1.3.10 (n-th group of Čech cohomology)
Let X, $\mathscr{F}$ and $\mathcal{U}$ as above. The n-th group of Čech cohomology of $\mathscr{F}$ with respect of the covering $\mathcal{U}$ *is the n-th group of cohomology of the complex $\big(C^p(\mathcal{U}, \mathscr{F}), d^p\big)$. We denote it by $\check{H}^n(\mathcal{U}, \mathscr{F})$.*

By definition of the maps res_j, we have a map

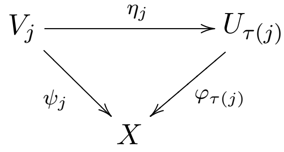
$$\mathscr{F}(X) \longrightarrow \check{H}^0(X, \mathscr{F}) = \ker\Big(\prod_i \mathscr{F}(U_i) \longrightarrow \prod_{i,j} \mathscr{F}(U_{i,j})\Big),$$

which is an isomorphism if $\mathscr{F}$ is a sheaf.

1.3.3.1 Getting rid of the dependence on the choice of the covering

The aim now is to get rid of the choice of the covering. In this section, a covering denotes an E-covering for some fixed class of morphism E.

Definition 1.3.11 (Refinement of a covering)
Let $\mathcal{U} = \big\{ U_i \xrightarrow{\varphi_i} X \big\}_{i \in I}$ and $\mathcal{V} = \big\{ V_j \xrightarrow{\psi_j} X \big\}_{j \in J}$ be two coverings of a scheme X. We say that $\mathcal{V}$ is a refinement *of $\mathcal{U}$ if there exists a map $\tau : J \longrightarrow I$ and morphisms $\eta_j : V_j \longrightarrow U_{\tau(j)}$ such that the following diagram commutes for every $j \in J$:*



If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then the family of morphisms $\{\eta_j\}$ induces a map of cochain complexes $\tau^\bullet : C^\bullet(\mathcal{U}, \mathscr{F}) \longrightarrow C^\bullet(\mathcal{V}, \mathscr{F})$ which induces maps on cohomology $\check{H}^n(\mathcal{U}, \mathscr{F}) \longrightarrow \check{H}^n(\mathcal{V}, \mathscr{F})$. Moreover, one can check the the maps between the cohomology groups do not depend on the choice of τ and η_j.

Consider now the category of all coverings $\mathcal{U}$ of X modulo the equivalence relation: $\mathcal{U} \sim \mathcal{V}$ if and only if $\mathcal{U}$ is a refinement of $\mathcal{V}$ and $\mathcal{V}$ is a refinement of $\mathcal{U}$. Then, being a refinement gives rise to a partial order on the equivalence classes of coverings of X. If we denote by $\mathcal{C}$ this diagram category, then $\mathcal{C}$ is a cofiltered category. Indeed, $\{U_i \times_X U_j \longrightarrow X\}$ is a refinement of both $\mathcal{U}$ and $\mathcal{V}$.

Definition 1.3.12 (n-th group of Čech cohomology)
*Let X be a scheme and $\mathscr{F}$ be a sheaf on X_E. The n-*th group of Čech cohomology of *$\mathscr{F}$ is defined as follows*

$$\check{H}^n(X_E, \mathscr{F}) = \varinjlim_{\mathcal{U} \in \mathcal{C}} \check{H}^n(\mathcal{U}, \mathscr{F}).$$

Remark 1.3.13
If $0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$ is an exact sequence of presheaves, then we get an exact sequence of complexes

$$0 \longrightarrow C^n(\mathcal{U}, \mathscr{F}') \longrightarrow C^n(\mathcal{U}, \mathscr{F}) \longrightarrow C^n(\mathcal{U}, \mathscr{F}'') \longrightarrow 0$$

which gives rise to a long exact sequence $\check{H}^n(\mathcal{U}, -)$ and, finally, a long exact sequence $\check{H}^n(X, -)$. However, if we start with an exact sequence of sheaves, then

$$0 \longrightarrow C^n(\mathcal{U}, \mathscr{F}') \longrightarrow C^n(\mathcal{U}, \mathscr{F}) \longrightarrow C^n(\mathcal{U}, \mathscr{F}'') \longrightarrow 0$$

may fail to be exact. In particular, we may have $H^n(X, \mathscr{F}'') \ncong \check{H}^n(X, \mathscr{F}'')$.

The previous remark motivates the following result (see [Mil80, Corollary III.2.5]).

Proposition 1.3.14
Let X be a scheme. Then, the Čech cohomology groups $\check{H}^n(X, -)$ agree with the cohomology groups $H^n(X, -)$ if and only if for every short exact sequence of sheaves, there is a functorially associated long exact sequence of Čech cohomology groups.

Proposition 1.3.15
Let X be a quasi-compact and quasi-projective scheme over an affine scheme. Then, we have isomorphisms $\check{H}^n(X_{\acute{e}t}, \mathscr{F}) \cong H^n(X_{\acute{e}t}, \mathscr{F})$ for every $n \in \mathbb{N}_0$ and every sheaf $\mathscr{F}$ on $X_{\acute{e}t}$.

Proof. See [Mil80, Theorem III.2.17]. □

1.3.3.2 Non-abelian Čech cohomology

As usual, let X be a scheme, X_E be a site on X and $\mathcal{U} = \{U_i \longrightarrow X\}$ be a covering of X for the E-topology. We want to extend the definition of $\check{H}^1(\mathcal{U}, \mathscr{F})$ to sheafs of non-abelian groups. Hence, we consider a sheaf of groups $\mathscr{F}$ on X_E. With a multiplicative notation, the usual definition gives.

Definition 1.3.16 (1-cocycle)
A 1-cocycle for $\mathcal{U}$ with values in $\mathscr{F}$ is given by an element $(f_{ij}) \in \prod_{i,j} \mathscr{F}(U_{i,j})$ which satisfies

$$f_{ij}\big|_{U_{i,j,k}} \cdot f_{jk}\big|_{U_{i,j,k}} = f_{ik}\big|_{U_{i,j,k}}, \quad \forall i, j, k.$$

Definition 1.3.17 (Cohomologous 1-cocycles)
Two 1-cocycles f and f' for $\mathcal{U}$ with values in $\mathscr{F}$ are said to be cohomologous *if there exists $(g_i) \in \prod_i \mathscr{F}(U_i)$ such that*

$$f'_{ij} = g_i\big|_{U_{ij}} \cdot f_{ij} \cdot g_j\big|_{U_{ij}}^{-1}, \quad \forall i, j.$$

Being cohomologous defines an equivalence relation on the set of 1-cocycles.

Definition 1.3.18 (First Čech cohomology group)
We denote by $\check{H}^1(\mathcal{U}, \mathscr{F})$ the set of equivalence classes of 1-cocycles.

Remark 1.3.19
If $\mathscr{F}$ is a sheaf of abelian groups, this definition coincides with the usual definition of $\check{H}^1(\mathcal{U}, \mathscr{F})$.

As in the commutative case, we define $\check{H}^1(X_E, \mathscr{F}) = \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, \mathscr{F})$.

1.3.3.3 Twisted forms and Čech cohomology

In this section, we fix a scheme X and a site X_E of schemes above X. As in section 1.1.5, the goal is to relate the twisted forms of an object to a first set of cohomology. Here, we will consider the twisted forms of an X-scheme with respect to the chosen E-topology.

Definition 1.3.20 (Twisted form of a scheme)
Let Y and Y' be two schemes over X. We say that Y' is a twisted form of Y for the E-topology *if there exists a covering $\{U_i \longrightarrow X\}_{i \in I}$ of X for the E-topology such that $Y \times_X U_i \cong Y' \times_X U_i$ for every $i \in I$.*

Notation 1.3.21
We denote by $\mathrm{TF}_E(Y)$ the set of isomorphism classes of twisted forms of Y for the E-topology.

Notation 1.3.22
Let Y be a X-scheme. We denote by $\underline{\mathrm{Aut}}(Y)$ the sheafification of the presheaf which associates to a X-scheme U the group $\mathrm{Aut}_U(Y \times_X U)$.

Proposition 1.3.23
We have a map of pointed sets

$$\theta : \mathrm{TF}_E(Y) \longrightarrow \check{H}^1(X, \underline{\mathrm{Aut}}(Y)).$$

Proof. Let Y' be a twisted form of Y for the E-topology. By hypothesis, there exists a covering $\mathcal{U} = \{U_i \longrightarrow X\}$ of X and isomorphisms $\phi_i : Y \times_X U_i \longrightarrow Y' \times_X U_i$ for each i. Now, we define

$$f_{ij} = \phi_i\big|_{U_{i,j}}^{-1} \circ \phi_j\big|_{U_{i,j}}$$

and it is easy to check that (f_{ij}) is indeed a 1-cocycle. If $\phi'_i : Y \times_X U_i \longrightarrow Y' \times_X U_i$ is another collection of isomorphisms which gives rise to the 1-cocycle (f'_{ij}), then f and

f' are cohomologous (via $g_i = \phi_i'^{-1} \circ \varphi_i$). Hence, the association $Y' \longmapsto (f_{ij})$ does not depend on the choice of the isomorphism. We thus have a map from $\mathrm{TF}_E(Y)$ to $\check{H}^1(\mathcal{U}, \mathrm{Aut}(Y))$ and thus to $\check{H}^1(X_E, \mathrm{Aut}(Y))$. Composing with the morphism of presheaves $\mathrm{Aut}(Y) \longrightarrow \underline{\mathrm{Aut}}(Y)$ gives a map from $\mathrm{TF}_E(Y)$ to $\check{H}^1(X, \underline{\mathrm{Aut}}(Y))$.
We want to check that if $Y' \cong Y''$ then their images are cohomologous. If we have an isomorphism $\phi : Y' \longrightarrow Y'$ and collections of isomorphisms $\phi_i : Y \times_X U_i \longrightarrow Y' \times_X U_i$ and $\phi_i' : Y \times_X U_i \longrightarrow Y'' \times_X U_i$ (if the two coverings are not the same, we can pass to the refinement $U_i \times_X U_j'$), then the images of Y and Y' are cohomologous via $g_i = \phi_i'^{-1} \circ \phi\big|_{U_i} \circ \phi_i$. □

Proposition 1.3.24
Suppose that the topology on X is one of the following: fpqc, fppf, étale, Zariski. Then, we have an injective map of pointed sets

$$\theta : \mathrm{TF}_E(Y) \longrightarrow \check{H}^1(X_E, \underline{\mathrm{Aut}}(Y)).$$

Proof. The additional assumption allows us to glue morphisms, which implies the injectivity of θ. □

Example 1.3.25
Let $Y = \mathcal{O}_X^n$. We have $\underline{\mathrm{Aut}}(Y) = \mathrm{GL}_n$. Then, we have an injection

$$\theta : \mathrm{TF}_{\text{ét}}(\mathcal{O}_X^n) \longrightarrow \check{H}^1(X_{\text{ét}}, \mathrm{GL}_n).$$

Chapter 2

The Brauer group of a field

The goal of this section is to present the construction of the Brauer group of a field, which can be used to classify central simple algebras which split over a field. We will see three constructions of this Brauer group. The first one, the "classical", uses Wedderburn's theorem. The second one express the Brauer group of a first cohomology group (non-abelian cohomology) while the last one is via a second group of cohomology; more precisely the Brauer group of a field k, is the cohomology group of the absolute Galois group with coefficient in k_s^*, the invertible elements of the separable closure of k.

Convention 2.0.26
When speaking about a k-algebra, it is assumed that k is a field. Moreover, unless stated otherwise, a k-algebra A is assumed to be finite dimensional.

2.1 Classical construction of the Brauer group

2.1.1 Preliminaries

Definition 2.1.1 (Splitting field)
Let A be a k-algebra. If K is a field extension of k such that $A \otimes_k K \cong \mathrm{M}_n(K)$, for some n, we say that K is a splitting field *for A. We may also say that A splits over K or that K splits A.*

Theorem 2.1.2 (Wedderburn's theorem)
Let A be a simple k-algebra. Then, there exists $n \in \mathbb{N}$ and a division algebra $k \subset D$ such that $A \cong \mathrm{M}_n(D)$. Moreover, the integer n is unique and D is unique up to isomorphism.

Corollary 2.1.3
Let k be an algebraically closed field and let A be a simple k-algebra. Then, there exists $n \in \mathbb{N}$ such that $A \cong \mathrm{M}_n(k)$.

Proof. Let n and D, given by Wedderburn's theorem, such that $A \cong \mathrm{M}_n(D)$. Consider some α in D. As D is finite dimensional over k, the powers $1, \alpha, \alpha^2, \ldots$ are linearly dependent. Therefore, there exists some polynomial $f \in k[x]$ such that $f(\alpha) = 0$. Since D is a division algebra, we may suppose that f is irreducible. Since k is algebraically closed, we have $\alpha \in k$, as required. □

Lemma 2.1.4
Let A be a k-algebra and K/k a finite field extension. Then, A is central simple (over k) if and only if $A \otimes_k K$ is central simple (over K).

We have the following characterization of central simple algebras:

Theorem 2.1.5
Let A be a k-algebra. Then, A is central simple if and only if there exists some finite field extension K of k such that A splits over K.

Proof. First, suppose that $A \otimes_k K \cong M_n(K)$. It is well-known that $M_n(K)$ is a central simple K-algebra and the previous lemma implies that A is central simple over k.
Now, suppose that A is a central simple k-algebra. The previous lemma and Corollary 2.1.3 imply that there exists an isomorphism $\varphi : A \otimes_k \overline{k} \longrightarrow M_n(\overline{k})$. We denote by $\alpha_1, \dots, \alpha_{n^2}$ the images under φ^{-1} of the canonical basis $e_1, \dots, e_{n^2}$ of $M_n(\overline{k})$. Now, there exists a finite field extension K of k such that $\alpha_1, \dots, \alpha_{n^2} \in A \otimes_k K$. The restriction of φ to $A \otimes_k K$ is an isomorphism between $A \otimes_k K$ and $\mathrm{M}_n(K)$, as required. □

Corollary 2.1.6
Let A be a k-algebra. Then, A is central simple if and only if there exists some finite Galois *extension K of k such that A splits over K.*

Corollary 2.1.7
If A is a central simple k-algebra, then its dimension (over k) is a square.

The last corollary motivates the following definition:

Definition 2.1.8 (Degree of a central simple algebra)
The degree *if a central simple k-algebra is $\sqrt{\dim_k A}$.*

The theorem also implies the following result:

Corollary 2.1.9
Let A_1, A_2 two central simple k-algebras. Then, $A_1 \otimes_k A_2$ is a central simple k-algebra. Moreover, if K is a splitting field for A_1 and A_2, then K is also a splitting field for $A_1 \otimes_k A_2$.

Proof. Let K_1 and K_2 be two finite extensions of k such that $A_i \otimes_k K_i \cong \mathrm{M}_{n_i}(K_i)$, for $i = 1, 2$. Then, the compositum field $K = K_1 \vee K_2$ is a splitting field for $A_1 \otimes_k A_2$. The second claim is obvious since $\mathrm{M}_{n_1}(K) \otimes_K \mathrm{M}_{n_2}(K) \cong \mathrm{M}_{n_1 \cdot n_2}(K)$. □

2.1.2 Construction of the Brauer group

Notation 2.1.10
Let $K|k$ be a finite Galois extension of fields. We denote by $\mathrm{CSA}_K(n)$ the set of k-isomorphism classes of central simple k-algebra which split over K.

Definition 2.1.11 (Brauer equivalent)
Let A, B be two central simple k-algebras. We say that A is Brauer equivalent *to B if there exists $m, n \in \mathbb{N}$ such that $A \otimes_k \mathrm{M}_m(k) \cong B \otimes_k \mathrm{M}_n(k)$*

One can easily show that being Brauer equivalent defines an equivalence relation.

Definition 2.1.12
As before, let $K|k$ be a finite Galois extension. We denote by $\mathrm{Br}(k)$ *the set of equivalence classes of central simple k-algebra (for the relation defined just above). We denote by* $\mathrm{Br}(K|k)$ *the set equivalence classes of central simple k-algebra (for the relation defined just above) which split over K.*

Theorem 2.1.13
The sets $\mathrm{Br}(k)$ *and* $\mathrm{Br}(K|k)$*, equipped with the tensor product (over k) are abelian groups.*

Proof. We only prove that $\mathrm{Br}(k)$ is a group and the Corollary 2.1.9 will imply that $\mathrm{Br}(K|k)$ is a subgroup of $\mathrm{Br}(k)$. It is easy to see that the tensor product induces a well-defined binary commutative operation on the equivalence classes. Moreover, the class of $k \cong \mathrm{M}_1(k)$ is the neutral element. Now, consider some central simple k-algebra A and denote by A^{op} the opposite algebra. Then, we have the isomorphism $A \otimes_k A^{\mathrm{op}} \cong \mathrm{M}_n(k)$, where n is the degree of A (recall that the degree is the square root of the dimension). Therefore, the class of A^{op} is the inverse of the class of A, as required. □

Definition 2.1.14 (Brauer group)
The group $\mathrm{Br}(k)$ *is called the* Brauer group of k. *The group* $\mathrm{Br}(K|k)$ *is the* Brauer group of k relative to K.

Remark 2.1.15
The association $K \longmapsto \mathrm{Br}(K)$ is a covariant functor from the category of fields to the category of abelian groups.

Sometimes, it is more convenient to work with the following equivalent definition of the equivalence relation: two central simple k-algebras A and B are equivalent if the two division algebras given by Wedderburn's theorem (see Theorem 2.1.2) are the same (up to k-isomorphism), that is if there exists $n, m \in \mathbb{N}$ and a division algebra $k \subset D$ such that $A \cong \mathrm{M}_n(D)$ and $B \cong \mathrm{M}_m(D)$.

Proposition 2.1.16 (Equivalent definition of the relation)
The two relations are equivalent.

Example 2.1.17 (Brauer group of an algebraically closed field)
The Corollary 2.1.3 implies that the Brauer group of an algebraically closed field is trivial.

Definition 2.1.18 (Quasi-algebraically closed field, C_1 field)
Let K be a field. We say that K is quasi-algebraically closed *(or C_1) if every non-constant homogeneous polynomial $p \in K[x_1, \ldots, x_n]$ of degree $d < n$ has at least one non-trivial zero in K^n.*

Example 2.1.19 (Brauer group of a quasi-algebraically closed field)
Let k be a quasi-algebraically closed field. We want to show that $\mathrm{Br}(k) = 0$, which is equivalent to prove that k is the unique division algebra over k. Let A be a division algebra of degree n over k and let $\{e_1, \ldots, e_{n^2}\}$ be a k-basis of A. Then, the reduced norm can be viewed as a homogeneous polynomial of degree d in n^2 variables over k:

$$\mathrm{Nrd} : k^{n^2} \longrightarrow k, \quad (x_1, \ldots, x_{n^2}) \longmapsto \mathrm{Nrd}\big(\sum_i x_i \cdot e_i\big).$$

Since A is a division algebra, the point (iv) of Proposition 1.2.2 implies that $n^2 \leq n$, which means that $A = k$, as required.

2.2 The Brauer group as a H^1

Proposition 2.2.1
Let K/k be a finite Galois extension K/k. We have a bijection of pointed set

$$\mathrm{CSA}_K(n) \cong H^1\big(\,\mathrm{Gal}(K,k), \mathrm{PGL}_n\,\big).$$

Proof. We want to apply Theorem 1.1.36. Let $V = \mathrm{M}_n(K)$ and let Φ the $(1,2)$-tensor which corresponds to the matrix multiplication. Since any automorphism of the matrix ring is inner, we have $\mathrm{Aut}_K(\Phi) \cong \mathrm{PGL}_n(K)$. Since any algebra can be viewed as a $(1,2)$-tensor, the Theorem 2.1.5 suggests that $\mathrm{CSA}_K(n) \cong \mathrm{TF}_K(\mathrm{M}_n(k), \Phi)$. Hence, we just have to check that if a $(1,2)$-tensor Ψ is such that Ψ_K satisfies the associative condition, then so does Ψ. The element $\Psi \in \mathrm{Hom}(W \otimes_k W, W)$ can be viewed as an element of $W \otimes_k W^* \otimes_k \otimes W^*$ as follows: if $\{e_i\}$ denotes a k-basis of W and if $\{\varepsilon^i\}$ denotes the corresponding dual basis, then

$$\Psi = \sum_{i,j,k} a_{ijk} e_k \otimes \varepsilon^i \otimes \varepsilon^j,$$

where $\Psi(e_i \otimes e_j) = \sum_k a_{ijk} e_k$. The condition of associativity of Ψ can be written as $\Psi \circ (\mathrm{id} \otimes \Psi) = \Psi \circ (\psi \circ \mathrm{id})$. Now, one can check that this condition is equivalent to

$$\sum_{k,m} a_{rsk} \cdot a_{ktm} = \sum_{k,m} a_{stk} \cdot a_{rkm}, \quad \forall r, s, t.$$

Now, if we do the same think for Ψ_K, it is easy to see that the associativity condition on Ψ_K implies the one on Ψ. Therefore, we have $\mathrm{CSA}_K(n) \cong \mathrm{TF}_K(\mathrm{M}_n(k), \Phi)$. Finally, we have the required bijection. $\square$

Now, we would like to get rid of the dependence on n. We wan to consider both of the families $\big\{\,\mathrm{CSA}_K(n)\big\}_n$ and $\big\{H^1\big(\mathrm{Gal}(K,k), \mathrm{PGL}_n(K)\big)\big\}_n$ as directed systems. Let n and m be positive integers. First, we define

$$\begin{aligned} \mu_{mn} : \mathrm{CSA}_K(m) &\longrightarrow \mathrm{CSA}_K(nm) \\ A &\longmapsto A \otimes_k \mathrm{M}_n(K). \end{aligned}$$

Now, suppose that $a : G \longrightarrow \mathrm{PGL}_m(K)$ represents the class of a 1-cocycle. From the (class of) the matrix a_σ we define a matrix $b_\sigma \in \mathrm{M}_{nm}(K)$, by putting n times a_σ along the diagonal and zeros outside of the diagonal, that is:

$$b_\sigma = \begin{pmatrix} a_\sigma & 0 & \dots & 0 & 0 \\ 0 & a_\sigma & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \dots & a_\sigma & 0 \\ 0 & 0 & \dots & 0 & a_\sigma \end{pmatrix}.$$

One can check that this gives rise to a map of pointed sets

$$\lambda_{mn} = H^1\big(\mathrm{Gal}(K,k), \mathrm{PGL}_m(K)\big) \longrightarrow H^1\big(\mathrm{Gal}(K,k), \mathrm{PGL}_{mn}(K)\big).$$

Now, consider the maps $\theta_n : \mathrm{CSA}_K(n) \longrightarrow H^1\big(G, \mathrm{PGL}_n(K)\big)$ as in the proof of Theorem 1.1.36. We claim that the maps θ_n form a morphism of directed system, which means that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{CSA}_K(m) & \xrightarrow{\theta_m} & H^1\big(\mathrm{Gal}(K,k),\mathrm{PGL}_m(K)\big) \\ \downarrow{\scriptstyle \mu_{mn}} & & \downarrow{\scriptstyle \lambda_{mn}} \\ \mathrm{CSA}_K(nm) & \xrightarrow{\theta_{nm}} & H^1\big(\mathrm{Gal}(K,k),\mathrm{PGL}_{nm}(K)\big). \end{array}$$

We compute:

$\lambda_{mn} \circ \theta_m$ Let $A \in \mathrm{CSA}_K(m)$ and let $f : \mathrm{M}_m(K) \longrightarrow A \otimes_k K$ be an isomorphism. The image of A under θ_m is the 1-cocycle a. Let $\sigma \in \mathrm{Gal}(K,k)$ and let $b_\sigma = \lambda_{mn}(a)$. If a_σ is viewed as an automorphism of $\mathrm{M}_m(K)$ then, the image of a matrix $M \in \mathrm{M}_{nm}(K)$ is obtained by applying a_σ to each of the n^2 blocs of size $m \times m$ of M.

$\theta_{nm} \circ \mu_{mn}$ We have $\mu_{mn}(A) = A \otimes_k \mathrm{M}_n(k)$ and we can use the isomorphism

$$g = f \otimes \mathrm{id}_{\mathrm{M}_n(k)} : \mathrm{M}_m(K) \otimes \mathrm{M}_n(k) \longrightarrow A \otimes_k K \otimes_k \mathrm{M}_n(k)$$

to compute the element $\theta_{nm} \circ \mu_{mn}(A)$. Looking at the isomorphism between $\mathrm{M}_{mn}(K)$ and $\mathrm{M}_m(K) \otimes \mathrm{M}_n(K)$ shows that $\theta_{nm} \circ \mu_{mn}(A) = \lambda_{mn} \circ \theta_m(A)$, as required.

Therefore, the functor $\varinjlim$ gives rise to a map

$$\theta : \varinjlim_m \mathrm{CSA}_K(m) \longrightarrow \varinjlim_m H^1\big(\mathrm{Gal}(K,k),\mathrm{PGL}_m(K)\big).$$

Now, remark that by definition of the direct limit, we have $\mathrm{Br}(K|k) = \varinjlim_n \mathrm{CSA}_K(n)$.

Notation 2.2.2
With the same notations as above, we write

$$H^1\big(\mathrm{Gal}(K,k),\mathrm{PGL}_\infty(K)\big) := \varinjlim_m H^1\big(\mathrm{Gal}(K,k),\mathrm{PGL}_m(K)\big).$$

Now, we would like to endow $H^1\big(\mathrm{Gal}(K,k),\mathrm{PGL}_\infty(K)\big)$ with the structure of a group. Let $n, m \in \mathbb{N}$ and define a map:

$$\mathrm{End}(K^n) \times \mathrm{End}(K^m) \longrightarrow \mathrm{End}(K^n \otimes_K K^m), \quad (\varphi,\psi) \longmapsto \varphi \otimes \psi.$$

Via the choice of a basis, this map restricts and corestricts to a map from

$$\mathrm{GL}_n(K) \times \mathrm{GL}_m(K) \longrightarrow \mathrm{GL}_{nm}(K).$$

Moreover, since a couple of a scalar matrix is sent to a scalar matrix, we have $\mathrm{PGL}_n(K) \times \mathrm{PGL}_m(K) \longrightarrow \mathrm{PGL}_{nm}(K)$. Finally, we get a well-defined map

$$\begin{aligned} H^1\big(G,\mathrm{PGL}_n(K)\big) \times H^1\big(G,\mathrm{PGL}_m(K)\big) &\longrightarrow H^1\big(G,\mathrm{PGL}_{nm}(K)\big) \\ (a,b) &\longmapsto a \cdot b, \quad (a \cdot b)_\sigma = a_\sigma \otimes b_\sigma. \end{aligned}$$

On the other hand, we have a binary operation

$$\begin{aligned} \mathrm{CSA}_K(n) \times \mathrm{CSA}_K(m) &\longrightarrow \mathrm{CSA}_K(nm) \\ (A,B) &\longmapsto A \otimes_K B \end{aligned}$$

and it is easy to check that these two laws are compatible with the maps λ_{nm}, λ_{mn}, μ_{nm}, μ_{mn}. Therefore, we have the following result:

Proposition 2.2.3
Let $K|k$ be a finite Galois extension. We have the following isomorphisms of groups

$$\mathrm{Br}(K|k) = \varinjlim_{n} H^1\big(\mathrm{Gal}(K,k), \mathrm{PGL}_n(K)\big).$$

Using Corollary 2.1.6 and taking the limit over all finite Galois extension of k gives the following result:

Proposition 2.2.4
Let k be a field and let k_s be the separable closure of k. We have the following bijection:

$$\mathrm{Br}(k) = H^1\big(\mathrm{Gal}(k_s,k), \mathrm{PGL}_\infty(k_s)\big).$$

2.3 The Brauer group as a H^2

Notation 2.3.1
Let k be a field. We denote by G_k the absolute Galois group of k, that is G_k is $\mathrm{Gal}(k_{\mathrm{s}}, k)$, where k_{s} denotes some separable closure of k.

Remark 2.3.2
If the characteristic of k is 0 or if k is a finite field, then $k_{\mathrm{s}} = \overline{k}$, some algebraic closure of k.

Notation 2.3.3
Let K/k be a Galois extension. We denote by $H^m(K/k)$ the m-th group of cohomology group $H^m(\mathrm{Gal}(K,k), K^*)$.

Theorem 2.3.4
Let k be a field and let K be a finite Galois extension of k. Then we have

$$Br(K|k) \cong H^2\big(\mathrm{Gal}(K,k), K^*\big), \quad \mathrm{Br}(k) \cong H^2(G_k, {k_s}^*).$$

Proof. Let $G = \mathrm{Gal}(K,k)$. First, consider the exact sequence

$$1 \longrightarrow K^* \longrightarrow \mathrm{GL}_n(K) \longrightarrow \mathrm{PGL}_n(K) \longrightarrow 1.$$

Using Hilbert's theorem 90 (see Theorem 1.1.26) and Proposition 1.1.25, we have an exact sequence

$$0 \longrightarrow H^1\big(G, \mathrm{PGL}_n(K)\big) \xrightarrow{\delta_n} H^2(G, K^*), \tag{2.1}$$

which gives rise to the following diagram

$$\begin{array}{ccccc} \mathrm{CSA}_K(m) & \xrightarrow{\theta_m} & H^1\big(G, \mathrm{PGL}_m(K)\big) & & \\ \Big\downarrow{\scriptstyle \mu_{mn}} & \circlearrowleft & \Big\downarrow{\scriptstyle \lambda_{mn}} & \searrow^{\delta_m} & H^2(G, K^*) \\ \mathrm{CSA}_K(nm) & \xrightarrow{\theta_{nm}} & H^1\big(G, \mathrm{PGL}_{nm}(K)\big) & \nearrow_{\delta_{mn}} & \end{array}$$

To see that $\delta_{nm} \circ \lambda_{mn} = \delta_m$, consider a representative $c : G \longrightarrow \mathrm{PGL}_m(G)$ of a class in $H^1(G, \mathrm{PGL}_m(K))$. Recall that a representative a of the element $\delta_m(c)$ can

be written $a_{\sigma,\tau} = b_\sigma \cdot {}^\sigma b_\tau \cdot {b_{\sigma\tau}}^{-1}$, where b_x is the preimage of c_x under the canonical surjection $\pi : \mathrm{GL}_m(K) \longrightarrow \mathrm{PGL}_m(K)$. Now, let $c' = \lambda_{mn}(c)$. Since c'_σ is n copies of c_σ along the diagonal, we can construct the b'_σ by putting n copies of b_σ along the diagonal. hence, we will have $\delta_m(c) = \delta_{nm} \circ \lambda_{mn}(c)$. Therefore, we can pass to the limit and get a morphism of groups $\delta : H^1(G, \mathrm{PGL}_\infty(K)) \longrightarrow H^2(G, K^*)$. Passing equation (2.1) to the limit shows that δ is injective.
To see that it is surjective, we will show that δ_n is surjective, where $n = [K : k]$. Let $c : G \times G \longrightarrow K^*$ the representative of a class in $H^2(G, K^*)$ and let V the n dimensional K-vector space of basis $\{e_\sigma : \sigma \in G\}$. We define n automorphisms of V:

$$\begin{aligned} e_\sigma : V &\longrightarrow V \\ \tau &\longmapsto a_{\sigma,\tau} \cdot e_{\sigma\tau}. \end{aligned}$$

We want to show that $a_{\sigma,\tau} = b_\sigma \cdot {}^\sigma b_\tau \cdot {b_{\sigma\tau}}^{-1}$ (this will also show that b is a 1-cocycle). Hence, we compute,

$$\begin{aligned} b_{\sigma\tau}(e_\eta) &= a_{\sigma\tau,\eta}\, e_{\sigma\tau\eta} \\ b_\sigma \circ {}^\sigma b_\tau(e_\eta) &= b_\sigma\big(\sigma(a_{\tau,\eta})\big) \cdot e_{\tau\eta} = {}^\sigma a_{\tau,\eta} \cdot a_{\sigma,\tau\eta}\, e_{\sigma\tau\eta}. \end{aligned}$$

Using the fact that a is a 2-cocycle gives the required equality. □

Thanks to Proposition 1.1.18, we get the following result.

Proposition 2.3.5
Let k be a field. Then Br(k) *is torsion.*

Example 2.3.6 (The Brauer group of $\mathbb{R}$)
We have $k = \mathbb{R}$, $k_\mathrm{s} = \mathbb{C}$ and $G_k = C_2$. With the notations of Example 1.1.5, we have $H^2(G_k, {k_\mathrm{s}}^*) = \left(\mathbb{C}^*\right)^{C_2} / \operatorname{im} m_\lambda$, where $m_\lambda : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ send a complex number z to $z \cdot \overline{z}$. Therefore, we have

$$\mathrm{Br}(\mathbb{R}) = H^2(G_k, {k_\mathrm{s}}^*) = \mathbb{R}^* / (\mathbb{R}^*)^2 \cong C_2.$$

Since the algebra $\mathbb{H}$ of Hamilton's quaternion is not isomorphic to some matrix algebra over $\mathbb{R}$, $\mathbb{R}$ and $\mathbb{H}$ are two representatives of the two classes of Br$(\mathbb{R})$.

Example 2.3.7 (The Brauer group of a finite field)
Let $k = \mathbb{F}_q$ be a finite field of characteristic p. Since k is a perfect field, we have $k_\mathrm{s} = \overline{k}$. The absolute Galois group of k is $G_k = \hat{\mathbb{Z}} = \varprojlim_n C_n$. Therefore, we have

$$\mathrm{Br}(k) = \varinjlim_{n \in \mathbb{N}_0} H^2\big(\mathrm{Gal}(\mathbb{F}_{q^n}, \mathbb{F}_q), {\mathbb{F}_{q^n}}^*\big).$$

Now, the $\mathrm{Gal}(\mathbb{F}_{q^n}, \mathbb{F}_q)$-fixed points of ${\mathbb{F}_{q^n}}^*$ are exactly the elements of ${\mathbb{F}_q}^*$. We see that the map $\mathrm{mult}_\lambda : {\mathbb{F}_{q^n}}^* \longrightarrow {\mathbb{F}_{q^n}}^*$ is the norm and is surjective onto the base field. Therefore, we have $H^2\big(\mathrm{Gal}(\mathbb{F}_{q^n}, \mathbb{F}_q), {\mathbb{F}_{q^n}}^*\big) = 1$, which implies that Br$(k) = 1$.

Remark 2.3.8
We could have used Wedderburn's little theorem which states that every finite skew field, or division algebra, is a field.

2.4 The Brauer group of a local field

The goal of this section is to present the computation of the Brauer group of a local field. We won't give all the details but instead present only the main steps. All the proofs can be found in [Ser95]. Let (K, v) be a complete valuation field. Let A be the valuation ring of K and let K_{res} the residue field of K. We suppose moreover that K_{res} is a *finite* field.

2.4.1 Main steps of the computation

To compute the Brauer group $\text{Br}(K)$ of K, the steps are the followings:

(i) Using the fact that $\text{Br}(K_{\text{ur}}) = 0$, deduce that for each central simple algebra $A \in \text{Br}(K)$, there exists a finite unramified extensions F/K such that F is a splitting field for K.

(ii) Last point implies that $\text{Br}(K) = H^2(K_{\text{ur}}/K)$. Hence, we have the equality $\text{Br}(K) = \bigcup_L H^2(L/K)$, where L runs through the set of finite unramified Galois extension of K contained in K_{s}. From now, we consider a finite unramified Galois extension L of K and its Galois group $G = \text{Gal}(L, K)$.

(iii) The split exact sequence

$$0 \longrightarrow U_L \hookrightarrow L^* \xrightarrow{v} \mathbb{Z} \longrightarrow 0$$

gives rise to split exact sequences

$$0 \longrightarrow H^m(G, U_L) \longrightarrow H^m(G, L^*) \xrightarrow{v} H^m(G, \mathbb{Z}) \longrightarrow 0$$

for every $m \in \mathbb{N}$.

(iv) Using the fact that $H^m(G, U_L) = 0$ for all $m \geq 1$, we get the isomorphism $H^m(G, L^*) \cong H^m(G, \mathbb{Z})$.

2.4.2 Computation of the Brauer group

Proposition 2.4.1
We have $\text{Br}(K_{\text{ur}}) = 0$.

Proof. See [Ser95, Chapter X, Examples of Fields with Zero Brauer Group]. □

By construction of K_{ur} (see Definition 1.2.4), we get a morphism of groups $\theta : \varinjlim \text{Br}(E) \longrightarrow \text{Br}(K_{\text{ur}})$, where E goes through the set of finite unramified extension E of K contained in K_{ur}. Since this morphism is easily seen to be injective, we have the following.

Proposition 2.4.2
For each central simple algebra A over K, there exists a finite unramified extension $E \subset K_{\text{ur}}$ of K which is a splitting field for A. This means that we have

$$\text{Br}(K) = \text{Br}(K_{\text{ur}}|K) = H^2(K_{\text{ur}}/K)$$

Proposition 2.4.3
We have $\text{Br}(K) = \varinjlim_L H^2(L/K)$, where L runs through the set of finite unramified Galois extension of K.

Proof. Let $A \in \mathrm{Br}(K)$ and let $F \subset K_{\mathrm{ur}}$ be an unramified splitting field for A. By Theorem 1.2.3 and Corollary 1.2.6 we can choose F to be Galois. □

From now on, we fix a finite unramified Galois extension L/K and let G be the Galois group $\mathrm{Gal}(L, K) = \mathrm{Gal}(L_{\mathrm{ur}}, K_{\mathrm{ur}})$. The aim is to compute $H^2(L/K)$. Consider the following exact sequence of groups

$$0 \longrightarrow U_L \hookrightarrow L^* \xrightarrow{v} \mathbb{Z} \longrightarrow 0.$$

Let $\pi \in L$ be a uniformising parameter π, which means that every $x \in L^*$ can be written in a unique ways as $x = \pi^n u$, where $n = v(x) \in \mathbb{Z}$ and $u \in U_L$. Since L is unramified over K, we can choose $\pi \in K$. Hence, the action on L^* can be restricted to U_L. Moreover, with this choice of an uniformizer, the last sequence is a *split* exact sequence of G-modules ($\mathbb{Z}$ is viewed as a trivial $\mathscr{G}$-module). Since the functor $H^m(G, -)$ commutes with direct sums, we have the following split exact sequence

$$0 \longrightarrow H^m(G, U_L) \longrightarrow H^m(G, L^*) \longrightarrow H^m(G, \mathbb{Z}) \longrightarrow 0.$$

Since $H^m(G, U_L) = 0$ for every $m \geq 1$ (see [Mil11]), we get $H^m(G, L^*) \cong H^m(G, \mathbb{Z})$. Now, the exact sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$ and the fact that the group $H^m(G, \mathbb{Q})$ is zero for $m \geq 1$ (see Proposition 1.1.6), gives the isomorphism $H^2(G, \mathbb{Z}) \cong H^1(G, \mathbb{Q}/\mathbb{Z})$. Since $\mathbb{Q}/\mathbb{Z}$ is a trivial G-module, we have the equality $H^1(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$. Until now, we have the following morphisms:

$$H^2(L/K) \xrightarrow[\cong]{v_*} H^2(G, \mathbb{Z}) \xrightarrow[\cong]{\delta^{-1}} H^1\big(G, \mathbb{Q}/\mathbb{Z}\big) = \mathrm{Hom}\big(G, \mathbb{Q}/\mathbb{Z}\big) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

Definition 2.4.4 (The invariant map)
The morphism $H^2(L/K) \longrightarrow \mathbb{Q}/\mathbb{Z}$ *is called the* invariant map *and is denoted by* $\mathrm{inv}_{L/K}$. *This map induces an isomorphism* $H^2(L/K) \longrightarrow \big(\frac{1}{[L:K]}\mathbb{Z}\big)/\mathbb{Z}$.

It is easy that all the morphisms of the above sequence are compatible with the morphisms of the directed system, that is: if $K \subset L \subset L' \subset K_{\mathrm{ur}}$ is another finite Galois unramified extension of K with Galois group G', then each square of the following diagram is commutative:

$$\begin{array}{ccccccccc}
H^2(L/K) & \xrightarrow{v_*} & H^2(G, \mathbb{Z}) & \xrightarrow{\delta^{-1}} & H^1\big(G, \mathbb{Q}/\mathbb{Z}\big) & = & \mathrm{Hom}\big(G, \mathbb{Q}/\mathbb{Z}\big) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \| \\
H^2(E/K) & \xrightarrow{v_*} & H^2(G', \mathbb{Z}) & \xrightarrow{\delta^{-1}} & H^1\big(G', \mathbb{Q}/\mathbb{Z}\big) & = & \mathrm{Hom}\big(G', \mathbb{Q}/\mathbb{Z}\big) & \longrightarrow & \mathbb{Q}/\mathbb{Z}.
\end{array}$$

Therefore, we can pass to the limit over L and get the following sequences of isomorphisms:

$$H^2(K_{\mathrm{ur}}/K) \xrightarrow{v_*} H^2(\mathrm{Gal}(K_{\mathrm{ur}}, K), \mathbb{Z}) \xrightarrow{\delta^{-1}} H^1\big(\mathrm{Gal}(K_{\mathrm{ur}}, K), \mathbb{Q}/\mathbb{Z}\big)$$

$$H^1\big(\mathrm{Gal}(K_{\mathrm{ur}}, K), \mathbb{Q}/\mathbb{Z}\big) = \mathrm{Hom}\big(\mathrm{Gal}(K_{\mathrm{ur}}, K), \mathbb{Q}/\mathbb{Z}\big) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

Note the followings:

Remarks 2.4.5 (i) $\mathrm{Hom}\big(\mathrm{Gal}(K_{\mathrm{ur}}, K), \mathbb{Q}/\mathbb{Z}\big)$ is the group of all *continuous* homomorphisms from $\mathrm{Gal}(K_{\mathrm{ur}}, K)$ to $\mathbb{Q}/\mathbb{Z}$, where $\mathbb{Q}/\mathbb{Z}$ is endowed with the discrete topology.

(ii) The isomorphism $\mathrm{Hom}\left(\mathrm{Gal}(K_{\mathrm{ur}}, K), \mathbb{Q}/\mathbb{Z}\right) \cong \mathbb{Q}/\mathbb{Z}$ is obtained via the choice of a topological generator for $\mathrm{Gal}(K_{\mathrm{ur}}, K)$, which is procyclic.

Definition 2.4.6 (The invariant map)
The isomorphism $H^2(K_{\mathrm{ur}}/K) \longrightarrow \mathbb{Q}/\mathbb{Z}$ *is called the* invariant map *and is denoted by* inv_K.

To summarize, we have the following theorem.

Theorem 2.4.7
Let (K, v) *be a complete valuation field. Let* A *be the valuation ring of* K *and let* K_{res} *the residue field of* K*. We suppose moreover that* K_{res} *is a finite field. Then,* $\mathrm{Br}(K) = \mathbb{Q}/\mathbb{Z}$.

Chapter 3

The Brauer-Grothendieck group

3.1 Azumaya algebra over a local ring and Brauer group of a local ring

In this section, R denotes a commutative local ring with maximal ideal $\mathfrak{m}$. The reference for this section is [Mil80].

Convention 3.1.1
Let S be a ring. An algebra A over S is assumed to be unital with S contained in its center.

Convention 3.1.2
Let A be an algebra over a ring R. By $\mathrm{End}_R(A)$, we mean the R-module consisting of all endomorphisms of A as an R-module.

Definition 3.1.3 (Azumaya algebra over a local ring)
Let A be an R-algebra. We say that A is an Azumaya algebra *over R if A is free of finite rank as a R-module and if the map*

$$\begin{aligned} A \otimes_R A^{\mathrm{op}} &\longrightarrow \mathrm{End}_R(A) \\ a \otimes a' &\longmapsto \varphi_{a,a'} : A \longrightarrow A, x \longmapsto axa', \end{aligned}$$

is an isomorphism of R-modules.

Proposition 3.1.4
Let A be an Azumaya algebra over R. Then, the center $Z(A)$ of A is R.

Proof. By definition, we have $R \subset Z(A)$. Now, let $c \in Z(A)$. Consider an R-basis $\{1 = a_1, \dots, a_n\}$ of A and the family of R-linear endomorphisms

$$\begin{aligned} \chi_j : A &\longrightarrow A \\ a_j &\longmapsto \delta^i_j. \end{aligned}$$

Note that since A is an Azumaya algebra, we have $\varphi(xc) = \varphi(x)c$ for every $x \in A$. Now, we get

$$c = \chi_1(1)c = \chi_1(c) = r_1 \in R,$$

as required. □

Proposition 3.1.5
Let A be an Azumaya algebra over R. There exists a bijection between the ideals of R and the ideals of A.

Proof. We have the following maps:

$$\begin{aligned} \{\text{ideals of } R\} &\longleftrightarrow \{\text{ideals of } A\} \\ J &\longmapsto JA \\ R \cap I &\longleftarrow I \end{aligned}$$

We want to check that these maps are inverse to each other. We keep the notations of the previous proof.

$IA \cap R = I$ The inclusion $I \subset IA \cap R$ is clear. Let $x = \sum_i r_i a_i \in IA \cap R$. By hypothesis, we have $r_i \in I$ for every i and since $x \in R$, we have $r_i = 0$ for every $i > 1$, as required.

$(J \cap R)A = J$ The inclusion $(J \cap R)A \subset J$ is clear. Now, suppose that $x \in J$ and write $x = \sum_i r_i a_i$, for some $r_i \in R$. For every i, we have $r_i = \chi_i(x)$. Now, we have $\chi_i(x) \in J$ since A is an Azumaya algebra, as required.

□

The last two propositions and the Skolem-Noether's theorem give the following.

Proposition 3.1.6
Let k be a field. An algebra A over k is central simple if and only if it is an Azumaya algebra.

Lemma 3.1.7
Let R be any commutative ring and let A be an R-algebra which is free of finite rank as an R-module. Let R' be a commutative R-algebra. Then, we have an isomorphism

$$\begin{aligned} \eta : \mathrm{End}_R(A) \otimes_R R' &\longrightarrow \mathrm{End}_{R'}(A \otimes R') \\ \varphi \otimes s &\longmapsto \eta(\varphi \otimes r) : x \otimes r' \longmapsto \varphi(x) \otimes rr'. \end{aligned}$$

Proof. Let n denotes the rank of A as an R-module. Both the domain and the codomain of η are free R'-modules of rank n^2. If we let $\varphi_{ij} \in \mathrm{End}_R(A)$ be the map which send a_k to $\delta_i^k a_j$, then a R'-basis for $\mathrm{End}_R(A) \otimes_R R'$ is $\mathcal{B} = \{\varphi_{ij} \otimes 1\}$ while a R'-basis for $\mathrm{End}_{R'}(A \otimes R')$ is given by $\mathcal{B}' = \{\varphi_{ij} \otimes \mathrm{id}_{R'}\}$. Then, it is easy that η maps $\mathcal{B}$ to $\mathcal{B}'$. □

Proposition 3.1.8 (*i*) *If A is an Azumaya algebra over R and R' is a commutative local R-algebra, then $A \otimes_R R'$ is an Azumaya algebra over R'.*

(*ii*) *Let A be an R-algebra which is free of finite rank as an R-module. If the algebra $\overline{A} = A \otimes_R R/\mathfrak{m}$ is an Azumaya algebra over $R/\mathfrak{m}$, then A is an Azumaya algebra over R.*

Proof. Let's keep the notation of the previous lemma. We have the following commutative diagram:

$$\begin{array}{ccc} (A \otimes_R A^{\mathrm{op}}) \otimes_R R' & \xrightarrow{\varphi \otimes \mathrm{id}_{R'}} & \mathrm{End}_R(A) \otimes_R R' \\ \cong \big\downarrow & & \cong \big\downarrow \eta \\ (A \otimes_R R') \otimes_{R'} (A \otimes_R R')^{\mathrm{op}} & \xrightarrow[\varphi']{} & \mathrm{End}_{R'}(A \otimes R'). \end{array}$$

Since $A \otimes_R A^{\mathrm{op}}$ is free as an R-module (hence faithfully flat), then φ is an isomorphism if and only if φ' is. This implies both statements. □

Corollary 3.1.9 (*i*) *If A and A' are Azumaya algebras over R, then so is $A \otimes_R A'$.*

(*ii*) *The matrix ring $\mathrm{M}_n(R)$ is an Azumaya algebra over R.*

Proof. (i) We have

$$A \otimes_R A' \otimes_R R/\mathfrak{m} \cong \left(A \otimes_R R/\mathfrak{m}\right) \otimes_{R/\mathfrak{m}} \left(A' \otimes_R R/\mathfrak{m}\right).$$

Using the point (*i*) of the previous proposition, we see that each term is an Azumaya algebra over $R/\mathfrak{m}$, that is a central simple algebra over $R/\mathfrak{m}$. Hence, the right hand side is a central simple algebra which implies, by the point (*ii*) of the previous proposition, that $A \otimes_R A'$ is an Azumaya algebra over R.

(ii) We use the point (*ii*) of the previous proposition and the fact that $\mathrm{M}_n(k)$ is a central simple algebra if k is a field.

□

Let A and A' be two Azumaya algebras over R. We say that A and A' are *similar* if there exists $n, n' \in \mathbb{N}$ such that $A \otimes_R \mathrm{M}_n(R) \cong A' \otimes_R \mathrm{M}_{n'}(R)$. Let $\mathrm{Br}(R)$ be the set of equivalence classes of Azumaya algebras. The tensor product endows $\mathrm{Br}(R)$ with the structure of an abelian monoid with identity element R. Since $A \otimes A^{\mathrm{op}} \sim R$, $\mathrm{Br}(R)$ is in fact an abelian group.

Definition 3.1.10 (Brauer group of a local ring)
Let R be a local ring. The group $\mathrm{Br}(R)$ *is called the* Brauer group *of R.*

Remark 3.1.11
It is obvious that if R is a field, then this definition coincides with our previous definition of the Brauer group.

3.2 The Brauer group of a scheme

In this section, X denotes a locally noetherian scheme.

Definition 3.2.1 (Azumaya algebra over a scheme)
An $\mathcal{O}_X$-algebra $\mathcal{A}$ is called an Azumaya algebra *over X if it is coherent (as a $\mathcal{O}_X$-module) and if $\mathcal{A}_x$ is an Azumaya algebra over $\mathcal{O}_{X,x}$ for every closed point x of X.*

Remarks 3.2.2 (i) If $\mathcal{A}$ is an Azuyama algebra, then A_x is an Azumaya algebra over $\mathcal{O}_{X,x}$ for every $x \in X$. To see this, we can assume without loss of generality that $X = \operatorname{Spec} R$ and that $\mathcal{A} \cong \widetilde{M}$ for some R-module M. Now, if x corresponds to the prime ideal $\mathfrak{p}$ and if $\mathfrak{p} \subset \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R, we have

$$M_{\mathfrak{p}} \cong M \otimes_R R_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} R_{\mathfrak{p}} \cong M_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} R_{\mathfrak{p}}.$$

The last term is an Azumaya algebra over $R_{\mathfrak{p}}$ by hypothesis on A and point (i) of Proposition 3.1.8.

(ii) Using Theorem I.2.9 of [Mil80], we see that $\mathcal{A}$ is locally free of finite rank as a $\mathcal{O}_X$-module.

Proposition 3.2.3
Let $\mathcal{A}$ be an $\mathcal{O}_X$-algebra that is of finite type as an $\mathcal{O}_X$-module. Then, the following are equivalent:

(i) $\mathcal{A}$ is an Azumaya algebra over X.

(ii) $\mathcal{A}$ is locally free as an $\mathcal{O}_X$-module and $A(x) := \mathcal{A}_x \otimes \kappa(x)$ is a central simple algebra over $\kappa(x)$ for every $x \in X$.

(iii) $\mathcal{A}$ is locally free as an $\mathcal{O}_X$-module and the canonical homomorphism

$$\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}^{\mathrm{op}} \longrightarrow \operatorname{End}_{\mathcal{O}_X}(\mathcal{A})$$

is an isomorphism.

(iv) There exists a covering $\{U_i \longrightarrow X\}$ for the étale topology on X such that for each i, there exists an r_i, for which $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \cong \mathrm{M}_{r_i}(\mathcal{O}_{U_i})$.

(v) There exists a covering $\{U_i \longrightarrow X\}$ for the flat topology on X such that for each i, there exists an r_i, for which $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \cong \mathrm{M}_{r_i}(\mathcal{O}_{U_i})$.

Proof. $(i) \Rightarrow (ii)$ The first part of the claim follows from the previous remark. The second part follows from Propositions 3.1.8 and 3.1.6.

$(ii) \Rightarrow (i)$ Follows from Proposition 3.1.8.

$(i) \Leftrightarrow (iii)$ We have the following isomorphisms of $\mathcal{O}_{X,x}$-algebras

$$\left(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}^{\mathrm{op}}\right)_x \cong \mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{A}_x{}^{\mathrm{op}}, \quad \left(\operatorname{End}_{\mathcal{O}_X}(\mathcal{A})\right)_x \cong \operatorname{End}_{\mathcal{O}_{X,x}}(\mathcal{A}_x).$$

Hence, the equivalence between (i) and (iii) follows from the definitions.

$(iv) \Rightarrow (v)$ By definition.

$(v) \Rightarrow (ii)$ Let $U = \coprod_i U_i$. Since each U_i is flat over X, then so is U which implies that $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_U$ is flat over X. Since $U \longrightarrow X$ is surjective, U is faithfully flat over X and thus $\mathcal{A}$ is flat over $\mathcal{O}_X$. Let $x \in X$. Since $\mathcal{A}_x$ is flat over $\mathcal{O}_{X,x}$, $\mathcal{A}_x$ is a free $\mathcal{O}_{X,x}$-module which means that there exists some open neighbourhood V of x such that $\mathcal{A}\big|_V$ is a free $\mathcal{O}_X\big|_V$-module. Hence, $\mathcal{A}$ is locally free, as required.
Finally, we have

$$\mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x} \cong \mathrm{M}_{r_i}(\mathcal{O}_{X,x})$$

and thus $\mathcal{A}(x) \cong \mathrm{M}_{r_i}(\kappa(x))$, as required.

$(i) \Rightarrow (iv)$ See [Mil80].

□

Remark 3.2.4
Using the point (ii) of the last proposition, it is easy to see that the tensor product of two Azumaya algebras over a scheme X is again an Azumaya algebra over X.

Let $\mathcal{A}$ and $\mathcal{A}'$ be two Azumaya algebras over X. We say that $\mathcal{A}$ and $\mathcal{A}'$ are *similar* if there exists two locally free $\mathcal{O}_X$-modules E and E' of finite rank such that

$$\mathcal{A} \otimes_{\mathcal{O}_X} \mathrm{End}_{\mathcal{O}_X}(E) \cong \mathcal{A}' \otimes_{\mathcal{O}_X} \mathrm{End}_{\mathcal{O}_X}(E').$$

Since $\mathrm{End}_{\mathcal{O}_X}(E) \otimes_{\mathcal{O}_X} \mathrm{End}_{\mathcal{O}_X}(E') \cong \mathrm{End}_{\mathcal{O}_X}\left(E \otimes_{\mathcal{O}_X} E'\right)$, this is an equivalence relation. Let $\mathrm{Br}(X)$ be the set of equivalence classes of Azumaya algebras over X. The tensor product endows $\mathrm{Br}(R)$ with the structure of an abelian monoid with identity element $\mathcal{O}_X$. Since $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \sim \mathcal{O}_X$, $\mathrm{Br}(X)$ is in fact an abelian group.

Example 3.2.5
Let R be a local noetherian ring and let $X = \operatorname{Spec} R$. The equivalence of categories between the category of finitely generated R-modules and the category of coherent $\mathcal{O}_X$-modules, we have $\mathrm{Br}(R) \cong \mathrm{Br}(\operatorname{Spec} R)$.

Definition 3.2.6 (Brauer group)
The abelian group $\mathrm{Br}(X)$ *is called the* Brauer group *of* X.

Remark 3.2.7 (Functoriality of Br)
Let $f : X \longrightarrow Y$ be a morphism of locally noetherian schemes and let $\mathcal{A}$ be an Azumaya over Y. Since X is locally noetherian, $f^*\mathcal{A}$ is a coherent $\mathcal{O}_X$-module. Moreover, Proposition 3.1.8.(i) implies that $\left(f^*A\right)_x$ is an Azumaya algebra over $\mathcal{O}_{X,x}$. It follows that Br is a contravariant functor from the category of locally noetherian schemes to the category of abelian groups.

3.3 The cohomological Brauer group of a scheme

Proposition 3.3.1
Let $\mathcal{A}$ be an Azumaya algebra on a scheme X and let φ an automorphism of $\mathcal{A}$. Then, φ is locally, for the Zariski topology, an inner automorphism, that is: there is a covering of X by open sets U_i and elements $a_i \in \mathcal{A}(U_i)$ such that $\phi\big|_{U_i}$ is given by the conjugation by a_i.

Proof. See [Mil80]. □

Definition 3.3.2
Let X be any scheme and let X_E be any site on X. We consider the followings prescheaves:

$$\begin{aligned}
\mathbb{G}_a &: (U \longrightarrow X) \longmapsto \Gamma(U, \mathcal{O}_U), \\
\mathbb{G}_m &: (U \longrightarrow X) \longmapsto \Gamma(U, \mathcal{O}_U)^*, \\
\mathrm{GL}_n &: (U \longrightarrow X) \longmapsto \mathrm{GL}_n(\Gamma(U, \mathcal{O}_U)), \\
\mathrm{PGL}_n &: (U \longrightarrow X) \longmapsto \mathrm{Aut}(\mathrm{M}_n(U)).
\end{aligned}$$

Proposition 3.3.3
The presheaves $\mathbb{G}_a$, $\mathbb{G}_m$, GL_n *and* PGL_n *are sheaves for the fppf topology (and thus for the Zariski and étale topologies).*

Proof. It is known that a presheaf $\mathscr{F}$ which is represented by a scheme (that is $\mathscr{F} \cong \mathrm{Hom}(-, X)$) is a sheaf for the fppf topology. The first three presheaves are represented by $\operatorname{Spec}\mathbb{Z}[x]$, $\operatorname{Spec}\mathbb{Z}[x, x^{-1}]$ and $\operatorname{Spec}\Big(\mathbb{Z}[T_{11}, \ldots, T_{nn}, T]/\langle T \cdot \det(T_{ij}) - 1\rangle\Big)$, where $\mathbb{Z}[T_{11}, \ldots, T_{nn}, T]$ is the polynomial ring with $n^2 + 1$ indeterminates over $\mathbb{Z}$ and $\det(T_{ij})$ is the determinant of the matrix whose coefficient (i, j) is T_{ij}. The claim is clear for the first two ones. For the third one, we consider a ring R and the morphisms

$$\mathrm{GL}_n(R) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathrm{Hom}\left(\left(\mathbb{Z}[T_{11}, \ldots, T_{nn}, T]/\langle T \cdot \det(T_{ij}) - 1\rangle\right), R\right).$$

The homomorphism Φ sends an invertible matrix M to the morphism

$$\begin{aligned} \varphi : \mathbb{Z}[T_{11}, \ldots, T_{nn}, T] &\longrightarrow R \\ T_{ij} \longmapsto M_{ij}, &\quad T \longmapsto \det(M)^{-1} \end{aligned}$$

which passes to the quotient. The homomorphism Ψ sends a morphism $\overline{\varphi}$ to the matrix M which satisfies $M_{ij} = \overline{\varphi}(T_{ij})$. For PGL_n, see [Mil80]. □

Proposition 3.3.4
The sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{GL}_n \longrightarrow \mathrm{PGL}_n \longrightarrow 1$$

is an exact sequence of sheaves for the Zariski and étale topologies.

Proof. Comes from Proposition 3.3.1. □

Definition 3.3.5 (Cohomological Brauer group of a scheme, Brauer-Grothendieck group of a scheme)
Let X be a scheme. The cohomological Brauer group *of X (or* Brauer-Grothendieck group *of X) is the group $H^2(X_{\acute{e}t}, \mathbb{G}_m)$. It is denoted by* $\mathrm{Br}'(X)$.

Example 3.3.6 ($X = \operatorname{Spec} k$)
Let k be a field, $X = \operatorname{Spec} k$ and let $G = \mathrm{Gal}(k_{\mathrm{s}}, k)$. We know that there is an equivalence of categories between the category of sheaves on $X_{\text{ét}}$ and the category of continuous G-modules (see Proposition 1.3.8). Moreover, if M is a continuous G-module and if $\mathscr{F}$ is the corresponding sheaf, then we have

$$H^0(G, M) = M^G \cong \mathscr{F}(X) \cong H^0(X_{\text{ét}}, \mathscr{F}).$$

Therefore, the étale cohomology agrees with the Galois cohomology (see [Rot08, Corollary 6.49]). Now, with the notation of Proposition 1.3.8, we have

$$M_{\mathbb{G}_m} = \varinjlim_{k'} \mathbb{G}_m(k') = \varinjlim_{k'} (k')^* = {k_{\mathrm{s}}}^*.$$

In particular, we have $\mathrm{Br}'\big(\operatorname{Spec} k\big) \cong \mathrm{Br}(k)$.

Proposition 3.3.7
Let X be a scheme. The set of isomorphism classes of Azumaya algebras of rank n^2 over X is isomorphic to $\check{H}^1\big(X, \mathrm{PGL}_n\big)$.

Proof. Using Proposition 3.2.3 and Proposition 1.3.24, we get an injection from the set of isomorphism classes of Azumaya algebras of rank n^2 over X into $\check{H}^1\big(X, \mathrm{PGL}_n\big)$. One can show that this map is in fact surjective. □

Theorem 3.3.8
Let X be a scheme. We have an injective homomorphism $\mathrm{Br}(X) \longrightarrow H^2(X_{ét}, \mathbb{G}_m)$.

Proof. See [Mil80, Theorem IV.2.5]. □

3.4 A brief summary

Let k be a field and let A be a finite dimensional k-algebra. Let X be a locally noetherian scheme and let $\mathcal{A}$ be an $\mathcal{O}_X$-algebra that is of finite type as an $\mathcal{O}_X$-module.

(i) A is a central simple k-algebra if and only if there exists $n \in \mathbb{N}$ and some finite separable extension K of k such that $A \otimes_k K \cong \mathrm{M}_n(K)$.
The $\mathcal{O}_X$-module $\mathcal{A}$ is an Azumaya algebra over $\mathcal{O}_X$ if and only if there exists a covering $\big\{U_i \longrightarrow X\big\}$ for the étale topology on X such that for each i there exists an r_i for which we have $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \cong \mathrm{M}_{r_i}(\mathcal{O}_{U_i})$.

(ii) A is a central simple k-algebra if and only if $A \otimes_k A^{\mathrm{op}} \cong \mathrm{End}_k(A)$. $\mathcal{A}$ is an Azumaya algebra over $\mathcal{O}_X$ if and only if $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}^{\mathrm{op}} \cong \mathrm{End}_{\mathcal{O}_X}(\mathcal{A})$.

(iii) The set of k-isomorphism classes of central simple k-algebras of dimension n^2 is isomorphic to $H^1\big(\mathrm{Gal}(k_{\mathrm{s}}, k), \mathrm{PGL}_n\big)$. The set of isomorphism classes of Azumaya algebras of rank n^2 over X is isomorphic to $\check{H}^1\big(X, \mathrm{PGL}_n\big)$.

(iv) We have

$$\mathrm{Br}(k) \xrightarrow{\cong} H^2\big(\mathrm{Gal}(k_{\mathrm{s}}, k), {k_{\mathrm{s}}}^*\big), \qquad \mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(X) := H^2(X_{\text{ét}}, \mathbb{G}_m).$$

(v) We have the isomorphisms

$$\mathrm{Br}(k) \cong \mathrm{Br}(\mathrm{Spec}\, k) \cong H^2\big(\mathrm{Gal}(k_{\mathrm{s}}, k), {k_{\mathrm{s}}}^*\big) \cong H^2\big((\mathrm{Spec}\, k)_{\text{ét}}, \mathbb{G}_m\big).$$

3.5 Some results

Proposition 3.5.1
If X is compact or if the number of connected components of X is finite, then the image of $\mathrm{Br}(X)$ in $\mathrm{Br}'(X)$ is torsion.

Proof. See [Groa, Corollary 1.5]. □

Theorem 3.5.2
Let X be a scheme. If then dimension of X is less or equal to one or if the dimension of X is 2 and X is regular, then $\mathrm{Br}(X) \cong \mathrm{Br}'(X)$.

Proof. See [Grob, Corollary 2.2]. □

Proposition 3.5.3
Let X be a regular integral scheme and let K be the field of rational function of X. Then, we have an injection $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(K)$.

Proof. The canonical map $\operatorname{Spec} K \longrightarrow X$ gives rise to a map

$$H^2(X_{\text{ét}}, \mathbb{G}_m) \longrightarrow H^2((\operatorname{Spec} K)_{\text{ét}}, \mathbb{G}_m) = \operatorname{Br}(K)$$

which is injective (see [Mil80, III.2.22]). The Theorem 3.3.8 allows us to conclude. □

Table of notations

$E \vee F$	The composite field of E and F
$H \leq_o G$	H is an open subgroup of G
$N \trianglelefteq_o G$	H is an open normal subgroup of G
Ab	Category of abelian groups
$\mathrm{Aut}_K(\Phi)$	Automorphism group of the tensor Φ
C_n	Cyclic group of order n
G_k	Absolute Galois group of the field k
$H^m(K/k)$	$H^m(\mathrm{Gal}(K,k), K^*)$
K_{s}	Separable closure of K
K_{ur}	Maximal unramified extension of K in K_{s}
$\mathbb{N}$	Set of postive integers $\{1, 2, \ldots\}$
$\mathbb{N}_0$	Set of non-negative integers $\{0, 1, 2, \ldots\}$
$\mathbb{P}$	Set of prime numbers
$P(X)$	Category of presheaves on X (with value in **Ab**)
${}_R\mathbf{Mod}$	Category of R-modules
$S(X)$	Category of sheaves on X (with value in **Ab**)
$\mathrm{TF}_E(Y)$	Set of twister forms of Y for the E-topology
$\mathrm{TF}_K(V, \Phi)$	Set of $K\|k$-twisted forms of (V, Φ)
X_E	Small E-site on X
$\mathbb{Z}_p$	Ring of the p-adic integers

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