ÉCOLE POLYTECHNIQUE
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SMA

Grothendieck topologies and schemes

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Contents

Abstract

The open subsets of the Zariski topology are often too big to apply some concepts of differential geometry to the theory of schemes. The notion of a Grothendieck topology (or site) on a category, which generalizes the open coverings of a topological space, enable us to work with finer topologies. The goal of this project is to introduce the concept of site and to apply it to the category of S-schemes. We will present different topologies on this and state some applications:

- the étale topology is related to Galois theory;
- the fpqc is useful to study some descent problem;
- the fppf topology allow us to study the representability of the relative Picard functor.

In this project, we use the following conventions:

- All rings are assumed to be commutative with unit.
- A compact topological space is what many authors call quasi-compact: when I speak about a compact topological space, I do not assume any Hausdorff condition.

1 Prerequisites

1.1 Fibred product and fiber of a morphism over a point

Definition 1.1 (Pullback)
Let $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ be two morphisms of a category $\mathscr{C}$. The pullback *of f and g (if it exists) is an object W and a pair of two morphisms $f' : W \longrightarrow Y$ and $g' : W \longrightarrow X$ of $\mathscr{C}$ such that $g \circ f' = f \circ g'$. Moreover, these morphisms must satisfy the following universal property: for every object W' of $\mathscr{C}$ and every pair of morphisms $f'' : W' \longrightarrow Y$ and $g'' : W' \longrightarrow X$ such that $f \circ g'' = g \circ f''$, there exists a unique morphism $h : W' \longrightarrow W$ which makes the following diagram commute:*

$$\begin{tikzcd}
W' \arrow[dr, dashed, "h"] \arrow[drr, bend left, "f''"] \arrow[ddr, bend right, "g''"'] & & \\
& W \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

We say sometimes that W is the product of X and Y over Z.

Definition 1.2 (Fibred product of two S-schemes)
Let X and Y be two S-schemes. Then, the fibred product *of X and Y over S, written $X \times_S Y$, is the pullback of X and Y over S.*

Remark 1.3
If $S = \operatorname{Spec} R$, we may write $X \times_R Y$ instead of $X \times_S Y$.

Proposition 1.4
Let $f : X \longrightarrow S$ and $g : Y \longrightarrow S$ be two S-schemes. Then, their fibred product over S exists.

Proof. Different steps of the construction:

(i) Affine case.

(ii) For some open subset U of X, use the product $X \times_S Y$ to construct $U \times_S Y$.

(iii) Construct $X \times_S Y$ from a collection $X_i \times_S Y$, where the X_i's form an open covering of X.

(iv) X and Y are any schemes and S is affine.

(v) Conclusion.

Details:

(i) First, we consider the case where X, Y and S are affine. Hence, we have some ring morphisms $\varphi : C \longrightarrow A$ and $\psi : C \longrightarrow A$. Thus, we can consider the tensor product $A \otimes_C B$ and the obvious maps $\tilde{\varphi} : B \longrightarrow A \otimes_C B$ and $\tilde{\psi} : A \longrightarrow A \otimes_C B$. Hence, we have the two following commutative squares:

$$\begin{tikzcd}
A \otimes_C B & B \arrow[l, "\tilde{\varphi}"'] \\
A \arrow[u, "\tilde{\psi}"] & C \arrow[l, "\varphi"] \arrow[u, "\psi"']
\end{tikzcd}
\xrightarrow{\mathrm{Spec}}
\begin{tikzcd}
\operatorname{Spec} A \otimes_C B \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\
\operatorname{Spec} A \arrow[r] & \operatorname{Spec} C
\end{tikzcd}$$

Using the properties of the tensor product and the equivalence of categories between the category of commutative unit rings and the categories of affine schemes gives the required result.

(ii) Suppose that we have the fibred product of X and Y over S and that $U \subset X$ is an open subset. We claim that $p_X^{-1}(U)$ is a fibred product for U and Y over S. Suppose we are given a scheme Z and two morphisms $\tilde{f} : Z \longrightarrow U$ and $\tilde{g} : Z \longrightarrow Y$ such that $g \circ \tilde{g} = f \circ i \circ \tilde{f}$ (where $i : U \longrightarrow X$ is the inclusion). We have the following diagram:

$$\begin{tikzcd}
Z \arrow[drr, bend left, "\theta"] \arrow[drrr, bend left, "\tilde{g}"] \arrow[ddr, bend right, "\tilde{f}"'] & & & \\
& p_X^{-1}(U) \arrow[r, hook] \arrow[d, "p_X\big|_{p_X^{-1}(U)}"] & X \times_S Y \arrow[r, "p_Y"'] \arrow[d, "p_X"] & Y \arrow[d, "g"] \\
& U \arrow[r, hook, "i"'] & X \arrow[r, "f"'] & S,
\end{tikzcd}$$

where θ is induced by the universal property of the fibred product. Since $\tilde{f}(Z) \subset U$, we have $\theta(Z) \subset p_X^{-1}(U)$. Therefore, the morphism θ factors through $p_X^{-1}(U)$, as required. It is clear that this morphism is unique.

(iii) Suppose we are given the fibred products $X_i \times_S Y$. We want to glue these schemes to obtain $X \times_S Y$. For i and j, denote by X_{ij} the intersection $X_i \cap X_j$ and by U_{ij} the preimage $p_{X_i}^{-1}(X_{ij})$ which is the fibred product of X_{ij} and Y over S, by the previous step. Now, U_{ji} is also the fibred product of X_{ij} and Y over S which means that we have an unique isomorphism $\varphi_{ij} : U_{ij} \longrightarrow U_{ji}$. Using the uniqueness of the map in the universal property, one can show that this isomorphisms are compatible. Hence, we can glue the schemes $X_i \times_S Y$ via the isomorphisms φ_{ij} to get a scheme P (we get the projections $p_X : P \longrightarrow Y$ and $p_Y : P \longrightarrow Y$ by gluing the projections from $X_i \times_S Y$ to X_i and Y).
Let Z be a scheme and let $\tilde{f} : Z \longrightarrow U$ and $\tilde{g} : Z \longrightarrow Y$ be two morphisms such that $g \circ \tilde{g} = f \circ \tilde{f}$. If we let $Z_i = \tilde{f}^{-1}(X_i)$, then we get a collection of morphisms

$$Z_i \longrightarrow X_i \times_S Y \hookrightarrow X \times_S Y$$

which are compatible with $\tilde{f}\big|_{Z_i} : Z_i \longrightarrow X_i$ and $\tilde{g} : Z \longrightarrow Y$. Gluing these morphisms gives the required morphism $\theta : Z \longrightarrow P$ which is compatible with $\tilde{f}$ and $\tilde{g}$. Moreover, if $\tilde{\theta} : Z \longrightarrow P$ is another compatible morphism, then we have $\theta_i = \tilde{\theta}_i$ which implies that $\theta = \tilde{\theta}$, as required. Hence, P satisfies the universal property of the fibred product of X and Y over S.

(iv) Using the previous step we can construct the fibred product of X and Y over S if S is affine.

(v) Let S_i be an affine covering of S and let $X_i = f^{-1}(S_i)$ and $Y_i = g^{-1}(S_i)$. The previous point implies that the fibred products $X_i \times_{S_i} Y_i$ exist. If Z is a scheme and if $\tilde{f} : Z \longrightarrow X_i$ and $\tilde{g} : Z \longrightarrow Y$ are two compatible morphisms (with $X_i \longrightarrow S$ and $Y \longrightarrow S$), then $\tilde{g}$ must factor through Y_i. Hence, $X_i \times_{S_i} Y_i$ satisfies the universal property of the fibred product of X_i and Y over S. Gluing the schemes $X_i \times_S Y$ (see *(iii)*) gives the required product $X \times_S Y$.

□

Example 1.5
Let $X = \mathbb{A}^n_K$, $Y = \mathbb{A}^m_K$ and $Z = \operatorname{Spec} K$. Then, we have

$$X \times_K Y \cong \operatorname{Spec}\left(K[x_1, \ldots, x_n] \otimes_K K[y_1, \ldots, y_m]\right) \cong \mathbb{A}^{n+m}_K.$$

Definition 1.6 (Base change)
Let $f : X \longrightarrow Y$ and $g : Y' \longrightarrow Y$ be two morphisms of schemes. We say that the morphism $X \times_Y Y' \longrightarrow Y'$ is the base change *of f by g.*

Definition 1.7 (Property stable under base change)
Let (P) be a property of morphisms of schemes. We say that (P) is stable under base change *if for each morphism $f : X \longrightarrow Y$ which satisfies (P), then every base change of f by a morphism g also satisfies (P).*

Examples 1.8
We will see that the followings properties are stable under base change:

(i) being flat (Proposition 2.20);

(ii) being surjective (Corollary 1.18);

(iii) being (locally) of finite type (Proposition 1.25);

(iv) being unramified or étale (Proposition 4.39);

(v) being quasi-compact (Proposition 5.5);

(vi) being surjective (Corollary 1.18);

(vii) being fpqc (Proposition 5.6).

Proposition 1.9
Let $X \longrightarrow Z$ and $Y \longrightarrow Z$ be two closed immersion into an affine space $Z = \operatorname{Spec} R$. Then, the underlying topological space of $X \times_Z Y$ is isomorphic with $X \cap Y$.

Proof. We know (see [Har77, Corollary II.5.9]) that there exists two ideals I and J of R such that $X \cong \operatorname{Spec} R/_I$ and $Y \cong \operatorname{Spec} R/_J$. Then, we have $X \times_Z Y \cong \operatorname{Spec}\left(R/_{(I+J)}\right)$, as required. □

Example 1.10
Let K be an algebraically closed field (with characteristic different from two). Take $X = \operatorname{Spec}\left(K[x,y]/_{\langle x^2+y^2-1\rangle}\right)$ (the unit circle), $Y = \operatorname{Spec}\left(K[x,y]/_{\langle x\rangle}\right)$ (the y-axis), and $Z = \mathbb{A}^2_K$. The number of closed points of intersection $X \cap Y$, should be two (points corresponding to $(0, \pm 1)$). We have

$$\begin{aligned} K[x,y]/_{\langle x^2+y^2-1\rangle} \otimes_{K[x,y]} K[x,y]/_{\langle x\rangle} &\underset{A.2}{\cong} K[x,y]/_{\langle x^2+y^2-1, x\rangle} \\ &\cong K[y]/_{\langle y^2-1\rangle} \cong K \times K, \end{aligned}$$

where A.2 denotes the Proposition A.2 of the appendix. Now, $\operatorname{Spec} K \times K$ has two closed points, as required.

Definition 1.11 (Fiber over a point)
Let $f : X \longrightarrow Y$ be a morphism of schemes and let $y \in Y$. We define the fiber over y *to be the scheme $X_y = X \times_Y \operatorname{Spec} k(y)$, where $k(y)$ denotes the residue field of y on Y and* $\operatorname{Spec} k(y) \longrightarrow Y$ *is the canonical morphism.*

Example 1.12
Consider the morphism $f : X = \operatorname{Spec} \mathbb{Q}[x,y]/\langle y^2 - x\rangle \longrightarrow \mathbb{A}^1_{\mathbb{Q}}$, induced by the obvious morphism $\mathbb{Q}[x] \hookrightarrow \mathbb{Q}[x,y] \longrightarrow \mathbb{Q}[x,y]/\langle y^2 - x\rangle$ (one can think of the projection of the parabola $y^2 = x$ on the x-axis). Let us compute some fibers:

Fiber over $t > 0$ We have:

$$\begin{aligned} X_{\langle x-1\rangle} &= X \times_{\mathbb{A}^1_{\mathbb{Q}}} \operatorname{Spec} \kappa(\langle x-1\rangle) \\ &\cong \operatorname{Spec}\left(\mathbb{Q}[x,y]/\langle y^2 - x\rangle \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]_{\langle x-t\rangle}/\mathbb{Q}[x]\mathbb{Q}[x]_{\langle x-t\rangle}\right) \\ &\cong \operatorname{Spec}\left(\mathbb{Q}[x,y]/\langle y^2 - x\rangle \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/\langle x-t\rangle\right) \\ &\underset{A.2}{\cong} \operatorname{Spec}\left(\mathbb{Q}[x,y]/\langle y^2 - x, x-t\rangle\right) \cong \operatorname{Spec}\left(\mathbb{Q}[y]/\langle y^2 - t\rangle\right) \\ &\cong \operatorname{Spec}(\mathbb{Q} \times \mathbb{Q}), \end{aligned}$$

where A.2 denotes the Proposition A.2 of the appendix. Therefore, the fiber over t has only two elements: two closed points corresponding to $(t, \pm\sqrt{t})$.

Fiber over 0 In a similar way, we find

$$X_{\langle 0\rangle} \cong \operatorname{Spec}\left(\mathbb{Q}[y]/\langle y^2\rangle\right).$$

The only prime ideal of $\mathbb{Q}[y]/\langle y^2\rangle$ is the $\langle y\rangle/\langle y^2\rangle$ one which corresponds to $y = 0$.

Example 1.13
Let $f : X = \operatorname{Spec} \mathbb{Z}[i] \longrightarrow Y = \operatorname{Spec} \mathbb{Z}$. We want to compute the fibers over different points of Y.

Fiber over the generic point We have

$$X_0 \cong \operatorname{Spec}\left(\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{Q}\right) \cong \operatorname{Spec} \mathbb{Q}[i].$$

Hence, the fiber over 0 has only one element.

Fiber over a prime p We have

$$X_{\langle p\rangle} \cong \operatorname{Spec}\left(\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{F}_p\right) \underset{A.2}{\cong} \left(\mathbb{F}_p[x]/\langle x^2 + 1\rangle\right).$$

Now, we consider three cases:

(i) If $p = 2$, then X_2 has only one point, corresponding to $\langle x+1\rangle/(x+1)^2$.
(ii) If $p \equiv 1 \bmod 4$, then $-1 = a^2$ and $X_{\langle p\rangle} \cong \operatorname{Spec} \mathbb{F}_p \times \mathbb{F}_p$.
(iii) If $p \equiv 3 \bmod 4$, then $x^2 + 1$ is irreducible in $\mathbb{F}_p$ and $X_{\langle p\rangle} \cong \operatorname{Spec} \mathbb{F}_p$.

Proposition 1.14
Let $f : X \longrightarrow Y$ be a morphism of schemes and $y \in Y$. Then, the projection $p : X \times_Y \operatorname{Spec} k(y) \longrightarrow X$ induces a homeomorphism from X_y to $f^{-1}(y)$.

Proof. See [Liu06, Proposition 3.1.16]. □

Proposition 1.15
Let $f : X \longrightarrow Y$ be a morphism of schemes and let $x \in X$ and $y = f(x)$. Then, we have $\mathcal{O}_{X_y,x'} \cong \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$, where $x' \in X_y$ corresponds to x.

Proof. We can consider the case where $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, $x = \mathfrak{p} \in \operatorname{Spec} B$ and $y = \mathfrak{q} \in \operatorname{Spec} A$. Then, $x' \in X_y$ correspond to the prime ideal $\mathfrak{p} \otimes_A \kappa(y)$ of $B \otimes_A \kappa(y)$ and we have

$$\mathcal{O}_{X_y,x'} \cong \big(B \otimes_A \kappa(y)\big)_{(\mathfrak{p}\otimes_A \kappa(y))} \cong B_{\mathfrak{p}} \otimes_{A_{\mathfrak{q}}} \kappa(y) \underset{A.2}{\cong} B_{\mathfrak{p}}/\mathfrak{q}B_{\mathfrak{p}} \cong \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}.$$

□

Lemma 1.16
Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. Points z of $X \times_S Y$ are in bijective correspondence to quadruples

$$(x, y, s, \mathfrak{p})$$

where $x \in X$, $y \in Y$, $s \in S$ are points with $f(x) = s$, $g(y) = s$ and $\mathfrak{p}$ is a prime ideal of the ring $\kappa(x) \otimes_{\kappa(s)} \kappa(y)$. The residue field of z corresponds to the residue field of the prime $\mathfrak{p}$.

Proof. See [Aut, Schemes, Lemma 17.5]. □

Corollary 1.17
Let $X \longrightarrow S \longleftarrow Y$ be two S-schemes. Let $x \in X$ and let $y \in Y$. Then, there exists $z \in X \times_S Y$ such that z is mapped to x and y under the projections if and only if x and y lie over the same point $s \in S$.

Corollary 1.18
Surjectivity is stable under base change.

Proof. Let $f : X \longrightarrow Y$ be a morphism of S-schemes and let $S' \longrightarrow S$ be a morphism of schemes. We have the following commutative diagram:

$$\begin{tikzcd}
X_{S'} \arrow[rr, "p"] \arrow[dr, "f_{S'}"'] \arrow[dd] & & X \arrow[d, "f"] \\
& Y_{S'} \arrow[r, "q"] \arrow[dl] & Y \arrow[d] \\
S' \arrow[rr] & & S,
\end{tikzcd}$$

where $X_{S'}$ is the fibred product of X and S' over S and $Y_{S'}$ is the fibred product of Y and S' over S. Let $Z \subset X$. By the previous Lemma, we have $q^{-1}\big(f(Z)\big) = f_{S'} \circ p^{-1}(Z)$. Taking $Z = X$ gives the surjectivity of $f_{S'}$. □

1.2 Morphisms (locally) of finite type, finite morphisms

Definition 1.19 (Morphism locally of finite type, morphism of finite type)
Let $f : X \longrightarrow Y$ be a morphism of schemes. We say that f is locally of finite type *if there exists some affine open covering $V_j = \operatorname{Spec} B_j$, $j \in J$, of Y such that $f^{-1}(V_j) = \bigcup_i \operatorname{Spec} A_{ij}$ for every j, where each A_{ij} is a finitely generated B_j-algebra. If in addition every $f^{-1}(V_j)$ can be covered by a finite number of such algebras, we say that f is of* finite type.

Definition 1.20 (Finite morphism)
Let $f : X \longrightarrow Y$ be a morphism of schemes. We say that f is finite *if there exists some affine covering $V_i = \operatorname{Spec} B_i$ of Y such that $f^{-1}(V_i)$ is affine equal to $\operatorname{Spec} A_i$ for every i and such that A_i is finitely generated as a B_i-module.*

Examples 1.21 (i) The projection of the plane $\mathbb{A}^2_K$ on $\mathbb{A}^1_K$ is of finite type but it is not finite.

(ii) The projection $f : \operatorname{Spec} K[x,y]/\langle x^2+y^2-1\rangle \longrightarrow \mathbb{A}^1_K$ is finite.

(iii) Let $K \subset L$ be two fields. Then, the morphism $\operatorname{Spec} L \longrightarrow \operatorname{Spec} K$ is finite if and only if L is a finite extension of K.

(iv) Let $f : X \longrightarrow Y$ be a morphism of finite type and $x \in X$. Then, the morphism $\operatorname{Spec} \mathcal{O}_{X,x} \longrightarrow \operatorname{Spec} \mathcal{O}_{Y,f(x)}$ is not necessarily of finite type. For example, $K[x]$ is of finite type over K but $K[x]_{\langle x\rangle}$ is not of finite type over K.

Proposition 1.22
Let $f : X \longrightarrow Y$ be a morphism of schemes. Then, f is locally of finite type if and only if for every open affine subset $V = \operatorname{Spec} B$ of Y, there exists a covering of $f^{-1}(V)$ consisting of affine subsets $U_i = \operatorname{Spec} A_i$ such that each A_i is a finitely generated B-algebra.

Proof. Let $V = \operatorname{Spec} B$ be an affine open subset of X. We know there exists some affine open covering $V_j = \operatorname{Spec} B_j$, $j \in J$, of Y such that $f^{-1}(V_j) = \bigcup_i \operatorname{Spec} A_{ij}$ for every j, where each A_{ij} is a finitely generated B_j-algebra. For each j, we choose an affine covering of $V \cap V_j$ by principal open subsets:

$$V \cap V_j = \bigcup_k V_{j,k}, \quad V_{j,k} = \operatorname{Spec}\big((B_j)_{b^j_k}\big), \quad b^j_k \in B_j.$$

Now, denote by $a^{i,j}_k$ the image of b^j_k in A_{ij}. Now, we have

$$f^{-1}\Big(\operatorname{Spec}\big((B_j)_{b^j_k}\big)\Big) = \operatorname{Spec}\big((A_{ij})_{a^{i,j}_k}\big)$$

and each $(A_{ij})_{a^{i,j}_k}$ is a finitely generated $(B_j)_{b^j_k}$-algebra. To summarize: we can cover V by open affine subsets $\tilde{B}_r$ such that each $f^{-1}(\tilde{B}_r)$ is covered by open affine subsets $\tilde{A}_{r,s}$ with $\tilde{A}_{r,s}$ a finitely generated $\tilde{B}_r$-algebra. Now, we have to show that all these $\tilde{A}_{r,s}$ are finitely generated B-algebra, which is just Lemma A.4.
The converse is clear. □

It is easy to prove the following similar statement for finite morphisms:

Proposition 1.23
Let $f : X \longrightarrow Y$ be a morphism of schemes. Then, f is locally of finite type if and only if for every open affine subset $V = \operatorname{Spec} B$ of Y, there exists a finite covering of $f^{-1}(V)$ consisting of affine subsets $U_i = \operatorname{Spec} A_i$ such that each A_i is a finitely generated B-algebra.

Corollary 1.24
Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two morphism of (locally) finite type. Then, $g \circ f$ is (locally) of finite type.

Proposition 1.25
Any base change of a morphism (locally) of finite type is (locally) of finite type.

Proof. Let $f : X \longrightarrow Y$ be a morphism locally of finite type and let $g : Z \longrightarrow Y$ be a morphism of schemes. We have to show that $p_Z : X \times_Y Z \longrightarrow Z$ is locally of finite type. We consider an open affine covering $Y_i = \operatorname{Spec} C_i$ of Y and we let $X_i = f^{-1}(Y_i)$ and $Z_i = g^{-1}(Y_i)$. We choose some open affine covering $Z_{ik} = \operatorname{Spec} B_{ik}$ of Z_i and $X_{ij} = \operatorname{Spec} A_{ij}$ of X_i (by hypothesis, each A_{ij} is a C_i-algebra of finite type). Using the properties of the fibred product, we see that $p^{-1}(Z_{ik}) = X \times_Y Z_{ik} \cong X_i \times_{Y_i} Z_{ik}$. Since $g \circ p_Z = f \circ p_X$, the set $p_Z^{-1}(Z_{ik})$ is covered by the open sets $X_{ij} \times_{Y_i} Z_{ik} \cong \operatorname{Spec}\big(A_{ij} \otimes_{C_i} B_{ik}\big)$. Since A_{ij} is a C_i-algebra of finite type, $A_{ij} \otimes_{C_i} B_{ik}$ is a B_{ik}-algebra of finite type.
If f is of finite type, then the number of X_{ij} is finite for each i and so the number of $A_{ij} \otimes_{C_i} B_{ik}$ is finite for each i and each k, which implies that p_Z is of finite type. □

The following Proposition gives an interesting property of finite morphisms.

Proposition 1.26
Let $f : X \longrightarrow Y$ be a finite morphism and $y \in Y$. Then, $f^{-1}(y)$ is finite.

Proof. Without loss of generality, we can suppose that $f : \operatorname{Spec} A \longrightarrow B$. The Proposition 1.14 implies that our claim is equivalent to show that X_y is finite. If we denote by $\mathfrak{q}$ the ideal corresponding to y, then we have $X_{\mathfrak{q}} = \operatorname{Spec} A \otimes_B k(\mathfrak{q})$. Since A is a finitely generated B-module, $X_{\mathfrak{q}}$ is a finitely generated $k(\mathfrak{q})$-module, that is a finite dimensional vector space. Hence, $A \otimes_B k(\mathfrak{q})$ is artinian and we use Proposition A.1 to see that $X_{\mathfrak{q}}$ is finite, as required. □

Remark 1.27
Even if the fibers of a morphism f are all finite, f might not necessarily be finite. For example, consider the morphism $f : \operatorname{Spec} \mathbb{C}[x, y]/\langle xy - 1\rangle \longrightarrow \operatorname{Spec} \mathbb{C}[x]$ (which can be viewed as the projection of the "graph" of $x \mapsto \frac{1}{x}$ on the x-axis). The different fibers are finite (in particular $X_{\langle x\rangle} = \emptyset$) but f is not finite, since $\mathbb{C}[x, y]/\langle xy - 1\rangle$ is not a finitely generated $\mathbb{C}[x]$-module. For the last assertion, we prove that we have $\mathbb{C}[x, y]/\langle xy - 1\rangle \cong \mathbb{C}\left[x, \frac{1}{x}\right]$. Let $\varphi : \mathbb{C}[x, y] \longrightarrow \mathbb{C}\left[x, \frac{1}{x}\right]$ the ring homomorphism which sends a polynomial $f(x, y)$ to $f\left(x, \frac{1}{x}\right)$. It is clear that φ is a surjective ring homomorphism. Furthermore, we have $\langle xy - 1\rangle \subset \ker \varphi$. For the other inclusion, we consider a polynomial $f(x, y)$ such that $f\left(x, \frac{1}{x}\right) = 0$. This means that we can write $f(x, y) = g(x, y)(xy - 1)$ with $g(x, y) \in k(x)[y]$. We write $g(x, y) = \frac{\tilde{g}(x,y)}{h(x)}$ with $\tilde{g}(x, y)$ primitive and $h(x)$ in $\mathbb{C}[x]$. Hence, we have $h(x) \cdot f(x, y) = \tilde{g}(x, y) \cdot (xy - 1)$. Now, h must divide the contents of $(xy - 1)$ but since $xy - 1$ is primitive, we have $h \in \mathbb{C}$. Finally, this implies that $f \in \langle xy - 1\rangle$ and thus $\mathbb{C}[x, y]/\langle xy - 1\rangle \cong \mathbb{C}\left[x, \frac{1}{x}\right]$, which is not a finitely generated $\mathbb{C}[x]$-module.

1.3 Dimension

1.3.1 Krull dimension

Definition 1.28 (Height of a prime ideal)
Let $\mathfrak{p}$ be a prime ideal of a ring R. The height of $\mathfrak{p}$, which is denoted by $\operatorname{ht} \mathfrak{p}$, is the supremum of $n \in \mathbb{N}$ such that there exists a chain of prime ideals of R

$$\mathfrak{p} = \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n.$$

Remarks 1.29 (i) We count the number of strict inclusions in a chain and not the number of prime ideals which appear.

(ii) If R is an integral domain, then we may have $\mathfrak{p}_n = 0$.

Definition 1.30 (Krull dimension of a ring)
The dimension, or Krull dimension*, of a ring R is the supremum of $\operatorname{ht} \mathfrak{p}$ taken over all primes $\mathfrak{p}$. We denote by $\dim R$ the Krull dimension of R.*

Remark 1.31
If $\mathfrak{p}$ is a prime ideal of R, we have $\dim R_{\mathfrak{p}} = \operatorname{ht} \mathfrak{p}$.

Examples 1.32 (i) The dimension of a field is 0. Conversely, if R is a domain which has dimension 0, then R is a field.

(ii) If K is a field, then the fact that $K[x]$ is a PID implies that $\dim K[x] = 1$. More generally, if R is a PID but not a field, then $\dim R = 1$.

(iii) If R is an artinian ring, then $\dim R = 0$ (see Proposition A.1).

(iv) Let K be a field. Since $K[x_1, \ldots, x_n]/\langle x_1, \ldots, x_k \rangle \cong K[x_{k+1}, \ldots, x_n]$, we have the following sequence of prime ideals

$$0 \subsetneq \langle x_1 \rangle \subsetneq \langle x_1, x_2 \rangle \subsetneq \ldots \subsetneq \langle x_1, \ldots, x_n \rangle.$$

Therefore, $\dim K[x_1, \ldots, x_n] \geq n$ (see next theorem for a better result).

Theorem 1.33
Let R be an integral domain which is a finitely generated algebra over K. Then, $\dim R$ is equal to the transcendental degree of R over K.

Proof. See [MR89, Theorem 5.6]. □

Corollary 1.34
Let k be a field. Then, we have $\dim k[x_1, \ldots, x_n] = n$.

1.3.2 Dimension of a topological space

Definition 1.35 (Dimension of a topological space)
Let X be a topological space. The dimension of X, denoted by $\dim X$, is the supremum of $n \in \mathbb{N}_0$ such that there exists a chain

$$Z_0 \subsetneq Z_1 \subsetneq Z_2 \subsetneq \ldots \subsetneq Z_n$$

of closed irreducible subsets of X.

Remarks 1.36 (i) $Z_0 \neq \emptyset$ since the empty set is not considered to be irreducible.

(ii) If X itself is irreducible, then we may have $Z_n = X$.

Example 1.37
If $X = \operatorname{Spec} K[x, y]$, we have the following chain of irreducible closed subsets: X, the "vertical line" and the point corresponding to $\langle x, y \rangle$. These closed and irreducible subsets correspond to the prime ideals 0, $\langle x \rangle$ and $\langle x, y \rangle$.

Definition 1.38 (Codimension of a closed irreducible subset)
Let X be a scheme and Z be a closed irreducible subset of X. We define the codimension *of Z in X, written* $\operatorname{codim}(Z; X)$, *as the supremum $n \in \mathbb{N}_0$ such that there exists a chain*

$$Z = Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n \subset X$$

of closed irreducible subsets.

Remarks 1.39 (i) Again, we count the number of strict inclusions instead of the number of irreducible closed subsets occurring in the chain.

(ii) Let $X = Y \sqcup Z$. Then, it is easy to see that $\dim X = \max\{\dim Y, \dim Z\}$.

Proposition 1.40
Let X be a topological space. Then:

(i) If Y is a subset of X endowed with the induced topology, then $\dim Y \leq \dim X$.

(ii) If $\{U_i\}_i$ is an open covering of X, then $\dim X = \sup_i \dim U_i$.

Proof. (i) If $Y_0 \subset Y_1 \subset \ldots \subset Y_n$ is a strictly increasing sequence of closed irreducible subsets of Y, then $\overline{Y_0} \subset \overline{Y_1} \subset \ldots \subset \overline{Y_n}$ is a strictly increasing sequence of closed irreducible subsets of X.

(ii) If Z is a closed irreducible subset of X, then $U \cap Z$ is a closed irreducible subset of U, if $U \cap Z \neq \emptyset$. Therefore, we have $\dim X \leq \dim U_i$ for every i. The point (i) gives the other inequality.

$\square$

Definition 1.41 (Dimension of a scheme)
The dimension of a scheme *is the dimension of its underlying topological space.*

Since we have an "inclusion-reversing" bijection between prime ideals of a ring R and closed irreducible subsets of $\operatorname{Spec} R$, via $\mathfrak{p} \longmapsto \mathcal{V}(\mathfrak{p})$, we have the following result:

Proposition 1.42
Let R be a ring. Then, $\dim R = \dim \operatorname{Spec} R$.

Example 1.43
Let $X = \operatorname{Spec} K[x,y]/\langle y - x^2 \rangle$. Then, we have the following sequence of prime ideals

$$0 \subsetneq \langle y - x^2 \rangle \subsetneq \langle x - 1, y - 1 \rangle.$$

Since $\dim \mathbb{A}^2_K = 2$, then $\dim X = 1$. In a similar way, we see that

$$\dim \operatorname{Spec} K[x,y,z]/\langle y - x^2 \rangle = 2.$$

Proposition 1.44
Let X be a scheme and Y be a closed irreducible set of generic point η. Then, we have $\dim \mathcal{O}_{X,y} = \operatorname{codim}(Y; X)$. *One equivalent formulation is that* $\dim \mathcal{O}_{X,x}$ *is equal to* $\operatorname{codim}\left(\overline{\{x\}}; X\right)$ *for every $x \in X$.*

Proof. First, we suppose that $X = \operatorname{Spec} R$ and we denote the point corresponding to η by $\mathfrak{p}_0$. We have:

(i) we have "inclusion-reversing" bijection between prime ideals of a ring R and closed irreducible subsets of $\operatorname{Spec} R$;

(ii) $\mathcal{V}(\mathfrak{p}) = \mathcal{V}(\mathfrak{q}) \Leftrightarrow \mathfrak{p} = \mathfrak{q}$ if $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals;

(iii) $\mathcal{V}(\mathfrak{p}_0) = Y$;

Using these three points, we find that $\dim R_{\mathfrak{p}_0} = \operatorname{ht} \mathfrak{p}_0 = \operatorname{codim}(Y; X)$. Now, suppose that X is an arbitrary scheme. Fix an affine neighbourhood of x and use intersection and closure to use the affine case. $\square$

Example 1.45
Let $X = \mathbb{A}^2_K$ and $Y = Y_0 = \{y\}$ for some closed point y of X. We know that the maximal ideal of $\mathcal{O}_{X,y}$ is the set of germs of functions which vanish at y. We can take a line passing through y for Y_1 and $Y_2 = \mathbb{A}^2_K$. Then, the corresponding ideals in $\mathcal{O}_{X,y}$ are the ideals generated by germs of functions which vanish on the line and the ideal 0 which corresponds to the zero polynomial.

2 Flatness

2.1 Flatness (for modules)

In this section, we use the approach of [Bou89].

Definition 2.1 (N-flat module)
Let M and N be two R-modules. We say that the module M is N-flat *if, for every injective morphism of R-modules $f : N' \longrightarrow N$, the corresponding morphism $\mathrm{id} \otimes f : M \otimes N' \longrightarrow M \otimes N$ is injective.*

Proposition 2.2
Let M and N be two R-modules. Suppose that for every finitely generated R-submodule $\tilde{N}$ of N the inclusion morphism $M \otimes \tilde{N} \longrightarrow M \otimes N$ is injective. Then, M is a N-flat module.

Proof. Let $f : N' \longrightarrow N$ be an injective morphism of R-modules. We can suppose that N' is a submodule of N and that f is just the inclusion. Now suppose that $x = \sum_{i=1}^{n} m_i \otimes n_i \in M \otimes N'$ is such that $\sum_i m_i \otimes n_i = 0$ in $M \otimes N$. We consider $\tilde{N} = \langle n_i : 1 \leq i \leq n \rangle_R$. By hypothesis, the composition of the two following morphisms is injective

$$M \otimes \tilde{N} \longrightarrow M \otimes N' \longrightarrow M \otimes N,$$

which implies that $x = 0$. □

Proposition 2.3
Let M be a N-flat module and $K \leq N$ be a submodule of N. Then:

- (i) *M is a K-flat module;*
- (ii) *M is a N/K-flat module.*

Proof. (i) This part is easy. Indeed, let $f : N' \longrightarrow K$ be a monomorphism. Then, the composition $N' \longrightarrow K \longrightarrow N$ is injective. Using the N-flatness, we find that the composition $M \otimes N' \xrightarrow{\mathrm{id} \otimes f} M \otimes K \longrightarrow M \otimes N$ is injective. Therefore, $\mathrm{id} \otimes f$ is also a monomorphism, as required.

(ii) Instead of considering any monomorphism, we suppose that $N'/K \longrightarrow N/K$ is the inclusion. Then, the following diagram is commutative and its two rows are exact

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "\mathrm{id}"'] & N' \arrow[r] \arrow[d, hook] & N'/K \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & K \arrow[r] & N \arrow[r] & N/K \arrow[r] & 0.
\end{tikzcd}$$

Then, we find

$$\begin{tikzcd}
 & M \otimes K \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & M \otimes N' \arrow[r, "g"] \arrow[d, hook, "b"] & M \otimes N'/K \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & M \otimes K \arrow[r, "f'"] & M \otimes N \arrow[r, "g'"] & M \otimes N/K &
\end{tikzcd}$$

The fact that f' is a monomorphism follows from point (i) and the fact that g is surjective follows from the right exactness of the functor $M \otimes -$. Since id is surjective and since b is injective (by point (i)), the snake lemma gives $\ker c = 0$, as required.

□

Proposition 2.4
Let M and $\{N_i\}_i$ be R-modules such that M is a N_i-flat module for each i. Then, M is a $\bigoplus M_i$-flat module.

Proof. See [Bou89, Lemma 5, paragraph 2, chapter I]. □

Definition 2.5 (Flat module)
Let M be an R-module. We say that M is flat *(over R) if M is a N-flat module for every R-module N, that is if every injective morphism of R-modules $f : N' \longrightarrow N$ gives rise to an injective map $\mathrm{id} \otimes f : M \otimes N' \longrightarrow M \otimes N$.*

Proposition 2.6
If M is a free R-module, then M is flat.

Proof. First, note that we can suppose that $M = R^{(I)} = \bigoplus_{i \in I} R$. Let $f : N \longrightarrow N'$ be an injective morphism of R-modules. We have $N \otimes M \cong N^{(I)}$ and $N' \otimes M \cong N'^{(I)}$ and the morphism corresponding to $f \otimes \mathrm{id}_M$ sends any element $\sum_i n_i$ to $\sum_i f(n_i)$. Hence, M is flat over R. □

Proposition 2.7
Let $\{M_i\}_{i\in I}$ and $\{N_i\}_{i\in I}$ denote two families of R-modules. For each $i \in I$, let $f_i : M_i \longrightarrow N_i$ be a morphism of R-modules. These morphisms induce a homomorphism of R-modules $f : \bigoplus_{i\in I} M_i \longrightarrow \bigoplus_{i\in I} N_i$, via $\sum_i x_i \longmapsto \sum_i f_i(x_i)$. Then, f is injective if and only if every f_i is injective.

Proposition 2.8
Let $\{M_i\}_i$ be a collection of R-modules. Then, $\bigoplus_i M_i$ is flat if and only if each M_i is flat.

Proof. Let $f : N \longrightarrow N'$ be a morphism of R-modules. We have the following commutative diagram

$$\begin{array}{ccc} \bigoplus_i M_i \otimes N & \longrightarrow & \bigoplus_i M_i \otimes N' \\ \cong \downarrow & & \downarrow \cong \\ \bigoplus_i \big(M_i \otimes N\big) & \longrightarrow & \bigoplus_i \big(M_i \otimes N'\big). \end{array}$$

The previous proposition allows us to conclude. □

Since every projective module is a direct summand of a free module, we get the following corollary.

Corollary 2.9
Let P be a projective module. Then P is flat.

Proposition 2.10
Let M be an R-module. Then, M is flat if and only if M is R-flat.

Proof. If M is flat, then the morphism $i : M \otimes I \longrightarrow M \otimes R$ is injective for every ideal I of R, by definition. Reciprocally, suppose that M is R-flat and consider an R-module N. We must show that M is N-flat. Since N can be written as $R^{(J)}/K$, for some set J, where $R^{(J)}$ denotes the sum $\bigoplus_{j\in J} R$, Propositions 2.3 and 2.4 imply that M is N-flat. Therefore, M is flat. □

Remark 2.11
Proposition 2.2 implies that it is sufficient to check that the canonical morphism $M \otimes I \longrightarrow M \otimes R$ is injective for every finitely generated ideal I.

Proposition 2.12
Let M be an R-module. Then:

(*i*) *If M is flat, then $M_{\mathfrak{p}}$ is a flat R-module for every $\mathfrak{p} \in \operatorname{Spec} R$.*

(*ii*) *$R_{\mathfrak{p}}$ is a flat R-module. More generally, if S is a multiplicative subset of R, then $S^{-1}R$ is a flat R-module.*

(*iii*) *If R is a PID, then M is flat if and only if M is torsion-free.*

(*iv*) *Let $R \longrightarrow S$ be a homomorphism of rings. If M is flat over R, then $M \otimes_R S$ is flat over S.*

(*v*) *If R is flat over a ring S and if M is flat over R, then M is flat over S.*

(*vi*) *If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of R-modules and if both M' and M'' are flat, then M is also flat (this is the reciprocal to Proposition 2.3).*

Proof. (*i*) Let $\mathfrak{p} \in \operatorname{Spec} R$ and $f : N \longrightarrow N'$ be an R-linear map. Then, we have

$$\begin{tikzcd} M_{\mathfrak{p}} \otimes N \arrow[r] \arrow[d, "\cong"'] & M_{\mathfrak{p}} \otimes N' \arrow[d, "\cong"] \\ M \otimes (R_{\mathfrak{p}} \otimes N) \arrow[r] & M \otimes (R_{\mathfrak{p}} \otimes N'). \end{tikzcd}$$

Since M is flat, the morphism $\mathrm{M} \otimes (R_{\mathfrak{p}} \otimes N) \longrightarrow M \otimes (R_{\mathfrak{p}} \otimes N')$ is injective and so is $M_{\mathfrak{p}} \otimes N \longrightarrow M_{\mathfrak{p}} \otimes N'$, as required.

(*ii*) Let M be an R-module and N be a submodule of N. Now, since we have $N \otimes_R S^{-1}R \cong S^{-1}N$ and $M \otimes_R S^{-1}R = S^{-1}M$ and since $S^{-1}N$ is a submodule of $S^{-1}M$, the morphism $N \otimes_R S^{-1}R \longrightarrow M \otimes_R S^{-1}R$ is injective, as required.

(*iii*) We know that M is flat if and only if the morphism $M \otimes I \longrightarrow M \otimes R \cong M$ is injective. Since R is a PID, M is flat if and only if M is torsion free.

(*iv*) & (*v*) Follows directly from the associativity of the tensor product.

(*vi*) We consider an injective morphism $f : N \longrightarrow N'$ and the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M' \otimes N \arrow[r] \arrow[d, hook] & M \otimes N \arrow[r] \arrow[d] & M'' \otimes N \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & M' \otimes N' \arrow[r] & M \otimes N' \arrow[r] & M'' \otimes N' \arrow[r] & 0. \end{tikzcd}$$

Then, the four lemma (special case of the five lemma) implies that the morphism $M \otimes N \longrightarrow M \otimes N'$ is injective, as required. □

Proposition 2.13 (Flatness and localization)
Let M be an R-module. Then, the followings are equivalent

(*i*) *M is flat over R.*

(*ii*) *$M_{\mathfrak{p}}$ is flat over $R_{\mathfrak{p}}$ for every $\mathfrak{p} \in \operatorname{Spec} R$.*

(*iii*) *$M_{\mathfrak{m}}$ is flat over $R_{\mathfrak{m}}$ for every $\mathfrak{m} \in \operatorname{MaxSpec} R$.*

Proof. The associativity of the tensor product implies (*i*) $\Rightarrow$ (*ii*). It is clear that (*ii*) implies (*iii*). Now, suppose that $f : N \longrightarrow N'$ is an injective R-linear map and let $K = \ker(\mathrm{id} \otimes M) : M \otimes N \longrightarrow M \otimes N'$. By hypothesis, we have $K_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m}$ of R. Therefore, Lemma A.3 implies that $K = 0$ and thus M is flat over R. □

Proposition 2.14
Let $\varphi : A \longrightarrow B$ a morphism of rings. Then, the followings are equivalent:

(*i*) *B is flat over A;*

(*ii*) *for each* $Q \in \operatorname{Spec} B$*, the module* B_Q *is flat over* $A_{\varphi^{-1}(Q)}$*;*

(*iii*) *for each* $Q \in \operatorname{MaxSpec} B$*, the module* B_Q *is flat over* $A_{\varphi^{-1}(Q)}$*.*

Proof. Again, it is clear that $(i) \Rightarrow (ii) \Rightarrow (iii)$. To show that (iii) implies (i), we consider some injective morphism of B-module $\psi : N \longrightarrow N'$. Then, we show that $\ker\big(\psi \otimes \mathrm{id} : N \otimes_A B \longrightarrow N' \otimes_A B\big) = 0$ as in the previous proof. □

Proposition 2.15
Let R be a local ring of maximal ideal $\mathfrak{m}$ and M be a finitely generated module over R. Then, the followings are equivalent:

(*i*) *M is free;*

(*ii*) *M is projective;*

(*iii*) *M is flat.*

Proof. We know already that the implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ hold. Hence, it remains to show that if M is flat, then M is free. We know that $M/_{\mathfrak{m}M}$ is a finite dimensional vector space over $R/_{\mathfrak{m}}$. We choose a basis $\overline{m}_1, \ldots, \overline{m}_n$ of it and elements $m_1, \ldots, m_n$ such that the image of m_i in the quotient $M/_{\mathfrak{m}M}$ is $\overline{m}_i$. Now, consider the following R-homomorphism

$$\varphi : R^{(n)} \longrightarrow M, \quad e_i \longmapsto m_i,$$

where e_i denotes the i-th vector of the canonical basis of $R^{(n)}$. Then, Nakayama's lemma implies that the map φ is surjective (see Proposition A.5). Let K denotes its kernel and consider the following exact sequence

$$0 \longrightarrow K \longrightarrow R^{(n)} \longrightarrow M \longrightarrow 0.$$

By hypothesis, the following sequence is also exact:

$$0 \longrightarrow K \otimes R/_{\mathfrak{m}} \longrightarrow R^{(n)} \otimes R/_{\mathfrak{m}} \longrightarrow M \otimes R/_{\mathfrak{m}} \longrightarrow 0.$$

Since the last two terms are isomorphic, we have $0 \cong K \otimes R/_{\mathfrak{m}} \cong K/_{\mathfrak{m}K}$. Since K is also finitely generated, using again Nakayama's lemma gives $K = 0$. Hence, φ is an isomorphism and thus M is free. □

2.2 Flatness (for schemes)

We know that a morphism $f : X \longrightarrow Y$ of schemes gives rise to a family of schemes parametrized by Y: for each $y \in Y$, we have the fiber $X_y = X \times_Y \operatorname{Spec} \kappa(y)$. If f is a flat morphism, then we get, in some sense, a family of schemes which varies continuously.

Definition 2.16 (Flat morphism of schemes)
Let $f : X \longrightarrow Y$ be a morphism of schemes and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. We say that $\mathscr{F}$ is flat over Y at a point $x \in X$ *if $\mathscr{F}_x$ is a flat $\mathcal{O}_{Y,y}$-module via the map $f_x^\sharp : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$.*
We say that $\mathscr{F}$ is flat over Y *if $\mathscr{F}$ is flat over Y at every $x \in X$. We say that X if* flat *over Y, or that f is flat, if $\mathcal{O}_X$ is flat over Y.*

Definition 2.17 (Faithfully flat morphism of schemes)
We say that a morphism of schemes $f : X \longrightarrow Y$ is faithfully flat *if it is flat and surjective.*

We have, indeed, the following result:

Proposition 2.18
Let $\varphi : R \longrightarrow S$ be a homomorphism of rings and $f : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ be the corresponding morphism of schemes. Then, φ is flat if and only if f is flat.

Remark 2.19
Let $f : X \longrightarrow Y$ be a morphism of schemes. The Proposition 2.13 implies that f is flat if and only if f is flat at x for every closed point $x \in X$.

Proposition 2.20
Let $f : X \longrightarrow Y$ be a morphism of schemes.

(i) *If f is an open immersion, then f is flat.*

(ii) *Let $g : Y \longrightarrow Z$ be a flat morphism of schemes and let $\mathbb{F}$ be an $\mathcal{O}_X$-module which is flat over Y. Then, $\mathbb{F}$ is flat over Z.*

(iii) *Flatness is stable under base change.*

(iv) *Let A, B be two rings and let $\varphi : A \longrightarrow B$ be a homomorphism. Denote by $f : X = \operatorname{Spec} B \longrightarrow Y = \operatorname{Spec} A$ the corresponding morphism of schemes. Let M be a B-module. Then, $\tilde{M}$ is flat over Y if and only if M is flat over A.*

Proof. (i) Indeed, $f_x^\sharp$ is an isomorphism for every x.

(ii) See (v) of Proposition 2.12.

(iii) We can reduce to the affine case and use (iv) of Proposition 2.12.

(iv) We have

$$\begin{aligned} \tilde{M} \text{ flat over } Y &\Leftrightarrow \left(\tilde{M}\right)_{\mathfrak{q}} \text{ flat over } B_{\mathfrak{q}}, \quad \forall \mathfrak{q} \in \operatorname{Spec} B \\ &\Leftrightarrow M_{\mathfrak{q}} \text{ flat over } B_{\mathfrak{q}}, \quad \forall \mathfrak{q} \in \operatorname{Spec} B \\ &\Leftrightarrow M \text{ flat over } A. \end{aligned}$$

Remark: for the last equivalence, one can proceed as in Proposition 2.14. □

Examples 2.21 (i) For $n > m$, projections from $\mathbb{A}^n$ onto $\mathbb{A}^m$ are flat.

(ii) The morphism of schemes $\operatorname{Spec} K \longrightarrow \operatorname{Spec} K[x]/\langle x^2\rangle$ is not flat. To see this, we have to show that the morphism of rings $\varphi : K[x]/\langle x^2\rangle \longrightarrow K$, which sends $\overline{f}$ to $f(0)$, is not flat. Since $x \cdot 1 = 0$, the point (iii) of Proposition 2.12 implies that φ is not flat.

(iii) The morphism $\operatorname{Spec} K[x, y]/\langle xy\rangle \longrightarrow \operatorname{Spec} K[x]$ induced by the obvious morphism is not flat since $K[x, y]/\langle xy\rangle$ is not torsion-free over $\operatorname{Spec} K[x]$. Note that this map corresponds to the projection of the "cross" $xy = 0$ to the affine line. Moreover, the fiber over 0 is $\mathbb{A}^1_K$, which is of dimension 1, while the fiber over another points contains just one point (dimension 0). The next proposition show that under some hypothesis, the dimension of the fibers must be constant.

Notation 2.22
Let X be a scheme and $x \in X$. We denote the Krull dimension of $\mathcal{O}_{X,x}$ by $\dim_x X$.

Proposition 2.23
Let $f : X \longrightarrow Y$ be a flat morphism of schemes of finite type over a field k. For any point $x \in X$, let $y = f(x)$. Then, we have

$$\dim_x(X_y) = \dim_x X - \dim_y Y.$$

Proof. See [Har77, Proposition III.9.5]. □

3 Grothendieck topology

3.1 Prerequisites of category theory

The goal of this section is to recall (or present) a few notions of category theory. We try to present the relate the introduced categorical concepts with the algebraic geometry.

If M is a smooth manifold, then we can recover the underlying set of M by considering the set $\mathrm{Hom}(\{\star\}, M)$, in the category of smooth manifolds. In a similar way, as we will see below, the underlying set of a group G is isomorphic to the set $\mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G)$. If we want to recover the underlying set of a scheme, over a fixed scheme is not sufficient. For example, if X denotes $\operatorname{Spec}\mathbb{C}[x,y]$, then the set $\mathrm{Hom}(\operatorname{Spec}\mathbb{C}, \operatorname{Spec}\mathbb{C}[x,y])$ is in bijection with the set of rings homomorphisms $\varphi : \mathbb{C}[x,y] \longrightarrow \mathbb{C}$. Such an homomorphism is determined by the images of x and y. Hence, we have the isomorphism $\mathrm{Hom}(\operatorname{Spec}\mathbb{C}, \operatorname{Spec}\mathbb{C}[x,y]) \cong \mathbb{C}^2$ (and we do not "catch" the points corresponding to the zero ideal or to irreducible curves induced by irreducible polynomials f). However, if we consider for a scheme X the sets $\mathrm{Hom}(Y, X)$, where Y is a scheme, then we will be able to recover all these informations. The association $Y \longmapsto \mathrm{Hom}(Y, X)$ is a general construction which is presented in the next notation.

Notation 3.1
Let $\mathscr{C}$ be a category and $A \in \mathscr{C}$ an object. We denote by $\mathrm{Hom}(A, -)$ the covariant functor which sends any $B \in \mathscr{C}$ to $\mathrm{Hom}(A, B)$ and any $f : B \longrightarrow B'$ to

$$\begin{aligned} \mathrm{Hom}(A, f) : \mathrm{Hom}(A, B) &\longrightarrow \mathrm{Hom}(A, B') \\ g &\longmapsto f \circ g. \end{aligned}$$

One can define the contravariant functor $\mathrm{Hom}(-, A)$ in a similar way. Sometimes, we will denote $\mathrm{Hom}(A, -)$ by h^A and $\mathrm{Hom}(-, A)$ by h_A.

Definition 3.2 (Functor of points)
If X is a scheme, the functor h_X is called the functor of points *of X.*

Definition 3.3 (Y-valued points of a scheme)
If X is a scheme, the elements of the set $h_X(Y) = \mathrm{Hom}(Y, X)$ are the Y-valued points *of X. If $Y = \operatorname{Spec} R$, we prefer to call the set $h_X(\operatorname{Spec} R)$ the R-valued points of X.*

Example 3.4
Let $X = \operatorname{Spec}\mathbb{Z}[x_1, \dots, x_n]/\langle f_1, \dots, f_m\rangle$ and R be any ring. The set $\mathrm{Hom}(\operatorname{Spec} R, X)$ is in bijection with morphisms of rings from $\mathbb{Z}[x_1, \dots, x_n]/\langle f_1, \dots, f_m\rangle$ to R, which are specified by tuples $(r_1, \dots, r_n) \in R^n$ such that $f_i(r_1, \dots, r_n) = 0$ for every $1 \le i \le m$. Therefore, for a ring R, the elements of

$$\big\{(r_1, \dots, r_n) \in R^n : f_i(r_1, \dots, r_n) = 0, \forall 1 \le i \le m\big\}$$

are the R-valued points of $\operatorname{Spec}\mathbb{Z}[x_1, \dots, x_n]/\langle f_1, \dots, f_m\rangle$.

Definition 3.5 (Representable functor)
Let $\mathscr{C}$ be a locally small category (i.e. a category in which the hom-sets $\mathrm{Hom}(A, B)$ are sets) and F a functor from $\mathscr{C}$ to **Set** *(the category of sets). We say that F is* representable *if there exists an object A of $\mathscr{C}$ and a natural isomorphism $\alpha : \mathrm{Hom}(A, -) \longrightarrow F$. If F is a contravariant functor, we say that F is* representable *if there exists a natural isomorphism $\alpha : \mathrm{Hom}(-, A) \longrightarrow F$.*

Examples 3.6 (i) Let $F : \mathbf{Grp} \longrightarrow \mathbf{Set}$ be the forgetful functor, where **Grp** denotes the category of groups. Then, we have $F \cong \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}, -)$, via the natural transformation

$$\begin{aligned} G &\longrightarrow \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G) \\ g &\longmapsto \phi_g : n \longmapsto g^n, \quad \forall g \in G, n \in \mathbb{Z}, \end{aligned}$$

for every group G.

(ii) Let $F : {}_R\mathbf{Mod} \longrightarrow \mathbf{Set}$ be the forgetful functor, where ${}_R\mathbf{Mod}$ denotes the category of R-modules. Then, we have a natural isomorphism of functors $F \cong \mathrm{Hom}_{{}_R\mathbf{Mod}}(R, -)$, via the natural transformation

$$\begin{aligned} M &\longrightarrow \mathrm{Hom}_{{}_R\mathbf{Mod}}(R, M) \\ m &\longmapsto \phi_m : r \longmapsto r \cdot m, \quad \forall m \in M, r \in R, \end{aligned}$$

for every R-module M.

(iii) Let R be a ring and let M, N be two R-modules. We consider the functor $F : {}_R\mathbf{Mod} \longrightarrow \mathbf{Set}$, from the category of R-modules to the category of sets, which sends an R-module L to the set $\mathrm{Hom}_R\big(N, \mathrm{Hom}_R(M, L)\big)$. Because of the tensor-hom adjunction, we have a natural isomorphism

$$\mathrm{Hom}_R\big(N, \mathrm{Hom}_R(M, L)\big) \cong \mathrm{Hom}_R\big(N \otimes_R M, L\big).$$

Hence, $N \otimes_R M$ represents the functor F.

Lemma 3.7 (Yoneda's lemma)
Let $\mathscr{C}$ be a locally small category and $F : \mathscr{C} \longrightarrow \mathbf{Set}$ be a functor. Then, for every $A \in \mathscr{C}$ we have $\mathrm{Nat}\big(\mathrm{Hom}_{\mathscr{C}}(A, -), F\big) \cong F(A)$, where $\mathrm{Nat}\big(\mathrm{Hom}_{\mathscr{C}}(A, -), F\big)$ denotes the set of natural transformations from $\mathrm{Hom}(A, -)$ to F.

Proof. We set

$$\begin{aligned} \Phi : \mathrm{Nat}\big(\mathrm{Hom}_{\mathscr{C}}(A, -), F\big) &\longrightarrow F(A) \\ \alpha &\longmapsto \alpha_A(\mathrm{id}_A). \end{aligned}$$

Moreover, we define

$$\begin{aligned} \Psi : F(A) &\longrightarrow \mathrm{Nat}\big(\mathrm{Hom}_{\mathscr{C}}(A, -), F\big) \\ a &\longmapsto \Psi(a), \end{aligned}$$

where the B-component of $\Psi(a)$ is

$$\begin{aligned} \Psi(a)_B : \mathrm{Hom}_{\mathscr{C}}(A, B) &\longrightarrow (B) \\ f &\longmapsto F\big(f\big)(a). \end{aligned}$$

A direct computation shows that this map is well defined, that is: the $\Psi(a)_B$ are the components of a natural transformation $\Psi(a)$. We get easily that these maps are inverse of each other, as required. $\square$

Remarks 3.8 (i) In particular, if B is an object of $\mathscr{C}$ and if $F = \mathrm{Hom}_{\mathscr{C}}(B, -)$, then we have

$$\mathrm{Nat}\big(\mathrm{Hom}_{\mathscr{C}}(A, -), \mathrm{Hom}_{\mathscr{C}}(B, -)\big) = \mathrm{Nat}\big(h^A, h^b\big) \cong \mathrm{Hom}_{\mathscr{C}}(B, A).$$

(ii) One can prove in a similar way that $\mathrm{Nat}\big(\mathrm{Hom}(-, A), F\big) \cong F(A)$ for every contravariant functor $F : \mathscr{C} \longrightarrow \mathbf{Set}$.

Notation 3.9 (Category of functors)
Let $\mathscr{C}$ be a category and $\mathscr{A}$ be any category. We denote by $\mathscr{A}^{\mathscr{C}}$ the category of functors from $\mathscr{C}$ to $\mathscr{A}$.

We are interested in Yoneda's lemma because of the following example.

Example 3.10 (The Yoneda embedding)
Let $\mathscr{C}$ denote some locally small category. In this case, we can consider the functor category $\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$ which objects are contravariant functors from $\mathscr{C}$ to **Set** and morphisms are natural transformations between them. Then, we can define a functor $G : \mathscr{C} \longrightarrow \mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$ which associates to every object A of $\mathscr{C}$ the functor $\mathrm{Hom}(A, -)$ and to every morphism $f : A \longrightarrow A'$ the usual natural transformation $\mathscr{G}(f) : \mathrm{Hom}(-, A) \longrightarrow \mathrm{Hom}(-, A')$. The Yoneda's lemma implies that G is fully faithful, that is G is an embedding of $\mathscr{C}$ into $\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$. Moreover, the contravariant representable functors from $\mathscr{C}$ to **Set** are the ones who lie in the image of G.

Example 3.11 (Functor of points)
Let $\mathfrak{Sch}$ be the categories of schemes. The Yoneda's lemma implies that $\mathrm{Hom}(-, X) : \mathfrak{Sch} \longrightarrow \mathbf{Set}^{\mathfrak{Sch}^{\mathrm{op}}}$, the functor of points of a scheme X, is fully faithful.

3.2 Grothendieck topology

In the definition of a sheaf (of abelian groups) we start by making a category $\mathscr{T}_X$ from a topological space X: the objects of $\mathscr{T}_X$ are the open sets of X and the morphisms are the inclusions between open sets. Since an open subset V of an open set U is equivalent to an inclusion morphism $V \longrightarrow U$, an open covering $\bigcup_{i \in I} U_i$ of a set U can be given by a collection of morphisms $\left\{ U_i \longrightarrow U \right\}_i$. We know the three following properties of open coverings:

(i) If U is an open set, then $\left\{ U \longrightarrow U \right\}$ is a covering of U.

(ii) If $V \subset U$ are open sets and $\left\{ U_i \longrightarrow U \right\}_i$ is a covering for U, then the collection $\left\{ U_i \cap V \longrightarrow V \right\}_i$ is an open covering of U. We remark that the intersection $U_i \cap V$ can be identified with the pullback (also called fibred product) $U_i \times_U V$.

(iii) If $\left\{ U_i \longrightarrow U \right\}_i$ and $\left\{ V_{ij} \longrightarrow U_i \right\}_j$ are coverings, then $\left\{ V_{ij} \longrightarrow U \right\}_{i,j}$ is a covering.

The previous observations motivate the following definition:

Definition 3.12 (Grothendieck topology, site)
Let $\mathscr{T}$ be a category. A Grothendieck topology*, or a* site*, on $\mathscr{T}$ is a collection $\mathrm{Cov}(\mathscr{T})$ of sets $\left\{ U_i \xrightarrow{\varphi_i} U \right\}_{i \in I}$ (here, we allow $I = \emptyset$) of morphisms of $\mathscr{T}$ called* coverings*. This collection must satisfy the following properties:*

(i) If $\varphi : V \longrightarrow U$ is an isomorphism, then $\{\varphi\} \in \mathrm{Cov}(\mathscr{T})$.

(ii) If $\left\{ U_i \longrightarrow U \right\}_i \in \mathrm{Cov}(\mathscr{T})$ and $\varphi : V \longrightarrow U$ is a morphism, then the fibred product $U_i \times_U V$ exists for every i and $\left\{ U_i \times_U V \longrightarrow V \right\}$ is a covering.

(iii) If $\left\{ U_i \xrightarrow{\varphi_i} U \right\}_i \in \mathrm{Cov}(\mathscr{T})$ and if $\left\{ V_{ij} \xrightarrow{\psi_{ij}} U_i \right\}_j \in \mathrm{Cov}(\mathscr{T})$ for every i, then $\left\{ V_{ij} \xrightarrow{\varphi_i \circ \psi_{ij}} U \right\}_{i,j}$ is also a covering.

A site is a pair $\big(\mathscr{T}, \mathrm{Cov}(\mathscr{T})\big)$.

Remarks 3.13 (i) We will often abuse the notations and denote a site by $\mathscr{T}$ instead of $\big(\mathscr{T}, \mathrm{Cov}(\mathscr{T})\big)$.

(ii) One can easily see that if X is a topological space and if $\mathscr{T}_X$ denotes its category, then the collection of open coverings (in the usual meaning) forms a site on $\mathscr{T}_X$.

Definition 3.14 (Morphism of topologies, morphism of sites)
Let $\mathscr{T}$ and $\mathscr{T}'$ be two topologies. A morphism of topologies *$f : \mathscr{T} \longrightarrow \mathscr{T}'$ is a functor such that:*

(*i*) *For every covering $\left\{ U_i \xrightarrow{\varphi_i} U \right\}_i$ of $\mathscr{T}$, the collection $\left\{ f(U_i) \xrightarrow{f(\varphi_i)} f(U) \right\}_i$ is a covering of $\mathscr{T}'$.*

(*ii*) *For every covering $\left\{ U_i \xrightarrow{\varphi_i} U \right\}_i$ of $\mathscr{T}$ and every morphism $V \longrightarrow U$ in $\mathscr{T}$, the morphism*

$$f(U_i \times_U V) \longrightarrow f(U_i) \times_{f(U)} f(V)$$

is an isomorphism for every i.

Remark 3.15
It is easy to see that the identity functor from a site to itself is a morphism of topologies. Moreover, a composition of two morphisms of sites is again a morphism of sites. Therefore, one can consider the category of sites and morphisms between them.

Example 3.16
Let X and Y denote two topological spaces and $\tilde{f} : X \longrightarrow Y$ be a continuous map. Then, we obtain a map of sites $f : \mathscr{T}_Y \longrightarrow \mathscr{T}_X$ by setting $f(V) = \tilde{f}^{-1}(V)$ for every open set V of Y. Furthermore, if $i : U \longrightarrow V$ is the inclusion between two open sets of Y, then we let $f(i)$ be the inclusion from $f^{-1}(U)$ to $f^{-1}(V)$.

Example 3.17
Let G be a topological group and let $\mathscr{C}$ be the category of continuous left G-sets. If $\{U_i\}_{i\in I}$ and U are objects of $\mathscr{C}$, we say that $\left\{ U_i \xrightarrow{\varphi_i} U \right\}$ is a covering of U if:

(i) each φ_i is continuous morphism of G-sets;

(ii) we have $\bigcup_{i\in I} \operatorname{im} \varphi_i = U$.

The fibred product of two continuous left G-sets $X \xrightarrow{\varphi} Z \xleftarrow{\psi} Y$ is taken in the category of sets: the underlying set of $X \times_Z Y$ is $\{(x, y) \in X \times Y : \varphi(x) = \psi(y)\}$ and $X \times_Z Y$ is endowed with the structure of a continuous G-set via

$$\begin{aligned} G \times \big(X \times_Z Y\big) &\longrightarrow \big(X \times_Z Y\big) \\ \big(g, (x, y)\big) &\longmapsto (g \cdot x, g \cdot y), \end{aligned}$$

which is well-defined since

$$\varphi(g \cdot x) = g \cdot \varphi(x) = g \cdot \psi(y) = \psi(g \cdot y),$$

and thus $(g \cdot x, g \cdot y) \in X \times_Z Y$ if $(x, y) \in X \times_Z Y$. It is easy to see that $\mathscr{C}$ with these coverings is a site. We denote this site by T_G.

Definition 3.18 (Presheaf)
Let $\mathscr{T}$ be a site and $\mathscr{C}$ an abelian category. A presheaf *on $\mathscr{T}$ with values in $\mathscr{C}$ is a contravariant functor from $\mathscr{T}$ to $\mathscr{C}$.*

Remark 3.19
Let $\mathscr{T}$ be a site, $\mathscr{C}$ be an abelian category and let F be a presheaf on $\mathscr{T}$. Let U be an object of $\mathscr{T}$ and $\left\{ U_i \xrightarrow{\varphi_i} U \right\}_{i\in I}$ be a covering of U. For every $i, j \in I$, the

canonical projections $U_i \times_U U_j \longrightarrow U_i$ and $U_i \times_U U_j \longrightarrow U_j$ induce morphisms $\eta_{ij} : F(U_i) \longrightarrow F(U_i \times_U U_j)$ and $\psi_{ij} : F(U_j) \longrightarrow F(U_i \times_U U_j)$ which give rise to morphisms $\eta_i : F(U_i) \longrightarrow \prod_j F(U_i \times_U U_j)$ and $\psi_i : F(U_i) \longrightarrow \prod_j F(U_i \times_U U_j)$. Taking the product over I gives rise to the following two morphisms

$$\prod_i F(U_i) \underset{\psi}{\overset{\eta}{\rightrightarrows}} \prod_{i,j} F\big(U_i \times_U U_j\big).$$

Definition 3.20 (Sheaf)
A sheaf *with values in* $\mathscr{C}$ *is a presheaf* F *with values in* $\mathscr{C}$ *such that for every covering* $\big\{U_i \xrightarrow{\varphi_i} U\big\}_i$ *the following diagram (see previous remark for the definition of* η *and* ψ*) is an equalizer*

$$F(U) \longrightarrow \prod_i F(U_i) \underset{\psi}{\overset{\eta}{\rightrightarrows}} \prod_{i,j} F\big(U_i \times_U U_j\big).$$

Remarks 3.21 (i) If T is a topological space, $\mathscr{T}$ is its category and $\mathscr{C} = \mathbf{Ab}$, the previous condition is equivalent to the local condition and the gluing condition of sheaves of abelian groups.

(ii) An abelian sheaf (or abelian presheaf) denotes a presheaf with values in **Ab** (or a sheaf with values in **Ab**).

(iii) Since equalizers and products in the category **Set**, we can consider presheaves and sheaves with values in **Set**.

Definition 3.22 (Morphism of presheaves, morphism of sheaves)
A morphism between two (pre)sheaves is a natural transformation between them.

Proposition 3.23 (Kernel of a morphism of presheaves)
Let F, G *be two presheaves on a site* $\mathscr{T}$ *and* $\alpha : F \longrightarrow G$ *be a morphism between them. Then, the kernel of* α *exists.*

Proof. For each object U in $\mathscr{T}$ we let $K(U) = \ker \alpha_U$ (recall that the component $\alpha_U : F(U) \longrightarrow G(U)$ is a morphism in an abelian category, so we may consider its kernel). Let $\varphi : V \longrightarrow U$ be a morphism in $\mathscr{T}$ and consider the commutative diagram:

$$\begin{tikzcd}
K(U) \arrow[r, "i_U"] \arrow[d, dashed] & F(U) \arrow[r, shift left, "0"] \arrow[r, shift right, "\alpha_U"'] \arrow[d, "F(\varphi)"] \arrow[dr, phantom, "\circlearrowleft"] & G(U) \arrow[d, "G(\varphi)"] \\
K(V) \arrow[r, "i_V"] & F(V) \arrow[r, shift left, "0"] \arrow[r, shift right, "\alpha_V"'] & G(V)
\end{tikzcd}$$

We find that $\alpha_V \circ F(\varphi) \circ i_U$ factors through 0. Hence, the universal property of the equalizer gives rise to a map $K(\varphi)$ from $K(U)$ to $K(V)$. Then, one can check that K is indeed a functor.
Now, if H is another presheaf and if $\beta : H \longrightarrow F$ is a morphism such that $\alpha \circ \beta = 0$, then all the components $\alpha_U \circ \beta_U$ are zero and the universal properties of the equalizers induce the components of a unique morphism $\gamma : H \longrightarrow K$, as required. □

Remark 3.24
If $\mathscr{T}$ is a small category, one can show that the presheaves on $\mathscr{T}$ form an abelian category.

Definition 3.25 (Sheaf associated to a presheaf, sheafification)
Let $\mathscr{T}$ *be a site and let* F *be a presheaf on* $\mathscr{T}$ *with values in some abelian category* $\mathscr{C}$*. Suppose there exists a sheaf* F^+ *on* $\mathscr{T}$ *with value in* $\mathscr{C}$ *and a morphism*

of presheaves $\alpha : F \longrightarrow F^+$ which satisfy the following universal property: for every sheaf G on $\mathscr{T}$ with values in $\mathscr{C}$ and every morphism of presheaves $\beta : F \longrightarrow G$, there exists a unique morphism of sheaves $\gamma : F^+ \longrightarrow G$ such that the following diagram commute:

$$\begin{tikzcd} F \arrow[r, "\beta"] \arrow[d, "\alpha"'] & G \\ F^+ \arrow[ur, dashed, "\gamma"'] & \end{tikzcd}$$

In this case, we say that F^+ is the sheaf associated *to F (or the sheafification of F).*

Remark 3.26
If the sheafification of a presheaf F exists, then it is unique up to isomorphism.

Theorem 3.27 (Sheaf associated to a presheaf)
Let $\mathscr{C}$ be a site and let F be a presheaf with values in **Ab** *or* **Set**. *Then, the sheafification of F exists.*

Proof. See [DG70, Theorem 4.3.14] and [Tam94, Theorem I.3.1.1]. □

Remark 3.28
For sheaves on topological spaces, the construction is detailed in [Har77, II.1.2].

3.3 Grothendieck topologies and schemes

To define the coverings of a scheme, we restrict ourselves to classes of morphisms E which satisfy the following conditions:

(i) all isomorphisms are in E;

(ii) E is closed under composition;

(iii) E is closed under base change.

Throughout this section, E denotes a class of morphism which satisfies the three conditions.

Example 3.29
The following classes of morphisms are stable under composition, base change and contain all isomorphisms:

- open immersions, $E = \text{zar}$;
- flat morphisms (see Proposition 2.20);
- étale morphisms (see Proposition 4.39), $E = \text{ét}$;
- morphisms (locally) of finite type (see Proposition 1.25).

Definition 3.30 (E-morphism)
A morphism which is in the class E is called an E-morphism.

Definition 3.31 (Slice category)
Let $\mathscr{C}$ be a category and $C \in \mathscr{C}$ an object of $\mathscr{C}$. The slice category of $\mathscr{C}$ over C *is the category which consists of:*

(*i*) *morphisms $B \xrightarrow{f_B} C$ in $\mathscr{C}$ as objects;*

(*ii*) *compatible morphisms as morphisms: if* $A \xrightarrow{f_A} C$ *and* $B \xrightarrow{f_B} C$ *are two morphisms in* $\mathscr{C}$*, a morphism between them is a morphism* $f : A \longrightarrow B$ *such that the following diagram commute:*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "f_A"'] & & B \arrow[dl, "f_B"] \\ & C & \end{tikzcd}$$

We denote this slice category by $\mathscr{C}/C$*.*

Let X be a scheme and $\mathscr{C}$ be a full subcategory of $\mathfrak{Sch}/X$ (this means that if $Y \longrightarrow X \longleftarrow Y'$ are two objects of $\mathscr{C}/X$ and if $f : Y \longrightarrow Y'$ is a compatible morphism in $\mathfrak{Sch}/X$, then f is a morphism in $\mathscr{C}/X$) which satisfies:

(i) $\mathscr{C}/X$ is closed under fibred products:

(ii) if $Y \longrightarrow X$ is an object of $\mathscr{C}/X$ and $V \longrightarrow Y$ is a morphism of E, then the composition $V \longrightarrow X$ is in $\mathscr{C}/X$.

Definition 3.32 (E-covering)
Let $Y \in \mathscr{C}/X$*. A family of* E*-morphisms* $\{V_i \xrightarrow{\varphi_i} Y\}$ *is called an* E-covering *of* Y *if* $\bigcup_i \varphi_i(V_i) = Y$*.*

Definition 3.33 (E-topology)
The class of all E*-coverings of all objects* Y *of* $\mathscr{C}/X$ *is called the* E-topology *on* $\mathscr{C}/X$*.*

Proposition 3.34
The category $\mathscr{C}/X$ *together with its* E*-topology is a site.*

Notation 3.35
We denote by $(\mathscr{C}/X)_E$, or X_E, the category $\mathscr{C}/X$ with its E-topology.

Definition 3.36 (Zariski site)
The (small) Zariski site*, denoted* X_{zar}*, is* $(\mathfrak{Sch}_{\mathrm{oi}}/X)_{zar}$*.*

Remark 3.37
If we consider the Zariski site X_{zar} and if we identify each open immersion $U \longrightarrow X$ with its image, we get the usual Zariski topology.

Example 3.38 (Constant presheaf)
Let A be an abelian group. To every object $U \longrightarrow X$ with $U \neq \emptyset$, we let $F(U) = A$. We also let $F(\emptyset) = 0$. If $\varphi : V \longrightarrow U$, with $V \neq \emptyset$, we set $F(\varphi) = \mathrm{id}_A$. It is clear that F is a presheaf on $X_{\text{ét}}$ which is called the *constant presheaf.* Note that this is not necessarily a sheaf. Consider for example the case where X is a non-connected topological space, say $X = U \cup V$ for two non-empty open sets U and V of X. Elements in $F(U)$ and $F(V)$ are mapped to zero in $F(U \cap V) = F(\emptyset) = 0$. Hence, if A is not trivial two distinct elements $a, b \in A$ won't satisfy the gluing axiom. One can show that if A is endowed with the discrete topology, then the sheafification F^+ of F (see Theorem 3.27) satisfy

$$F^+(U) = \{f : U \longrightarrow A : f \text{ continuous}\},$$

for each open set U of X. In particular, if some open set U is connected, then $F^+(U) = A$.

3.4 Pullback

Let $f : X \longrightarrow Y$ be a morphism of schemes. The morphism f induces a functor from the category of étale Y-schemes to the category of étale X-schemes: an Y-scheme Y' is sent to $X \times_Y Y'$. Now, suppose that $g : Y' \longrightarrow Y''$ is a morphism of Y-schemes. Then, we have the following commutative diagram:

$$
\begin{tikzcd}
X \times_Y Y' \arrow[rrr] \arrow[dr, dashed, "f_{Y'}"] \arrow[ddr, bend right, "p'_X"'] & & & Y' \arrow[dl, "g"] \arrow[ddl, bend left] \\
& X \times_Y Y'' \arrow[r] \arrow[d, "p_X"] & Y'' \arrow[d] & \\
& X \arrow[r, "f"'] & Y &
\end{tikzcd}
$$

Since the structural morphisms of Y' and Y'' are étale, then so are p_X and p'_X. Since $p'_X = p_X \circ f_{Y'}$, $f_{Y'}$ is étale (see Proposition 4.42). Hence, we get a covariant functor from the category of étale Y-schemes to the category of étale X-schemes.

Now, one can check that f induces a morphism of topologies (see Definition 3.14).

4 Étale topology

4.1 (Co)tangent space

Let M be a real smooth manifold of dimension n. Let $p \in M$ and let T_pM denote the tangent space of M at p. Let $\mathcal{O}_p$ be the ring of germs of smooth functions around p and let $\mathfrak{m}_p$ be its maximal ideal, which consists of class of functions vanishing at p. A derivative $v \in T_pM$ gives rise to a map $\mathfrak{m}_p \longrightarrow \mathbb{R}$: a smooth function f is mapped to its derivative $v[f]$. Since $v[f \cdot g] = v[f] \cdot g(p) + f(p) \cdot v[g] = 0$, this map gives rise to an $\mathbb{R}$-linear form $\Phi(v) \in \mathrm{Hom}\left(\mathfrak{m}_p/\mathfrak{m}_p^2, \mathbb{R}\right)$. Hence, we have an $\mathbb{R}$-linear map $\Phi : T_pM \longrightarrow \mathrm{Hom}\left(\mathfrak{m}_p/\mathfrak{m}_p^2, \mathbb{R}\right)$. This map is clearly injective. For the surjectivity, we fix a basis $\left\{\partial_{x^1}, \dots, \partial_{x^n}\right\}$ of T_pM and the dual basis $\left\{dx^1, \dots, dx^n\right\}$. Now, if α is an element of $\mathrm{Hom}\left(\mathfrak{m}_p/\mathfrak{m}_p^2, \mathbb{R}\right)$, then we let $v_i = \alpha(dx^i)$ and $v = \sum_{i=1}^n v_i \partial_{x^i}$. This vector v satisfies $\Phi(v) = \alpha$. Therefore, we have an isomorphism of $\mathbb{R}$-vector spaces $T_pM \cong \mathrm{Hom}\left(\mathfrak{m}_p/\mathfrak{m}_p^2, \mathbb{R}\right)$.

This motivates the following definition:

Definition 4.1 (Zariski tangent space, Zariski cotangent space)
Let X be a scheme and $x \in X$. The Zariski cotangent space *at x is the $\kappa(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. The* Zariski tangent space *is the dual of the cotangent space.*

Remark 4.2
Let $f : X \longrightarrow Y$ be a morphism of schemes and let $x \in X$ and $y = f(x)$. The morphism $f_x^\sharp$ restricts and corestricts to a morphism $\mathfrak{m}_y \longrightarrow \mathfrak{m}_x$, which gives rise to a morphism $\mathfrak{m}_y/\mathfrak{m}_y^2 \longrightarrow \mathfrak{m}_x/\mathfrak{m}_x^2$. Hence, a morphism of schemes induces a map on the cotangent spaces.

4.2 Differentials

4.2.1 Module of relative differential forms

Definition 4.3
Let M be a B-module where B is an A-algebra. We say that $d : B \longrightarrow M$ is an A-derivation *of B into M if d satisfies:*

(i) *d is additive;*

(ii) *$d(a) = 0$, for all $a \in A$;*

(iii) *$d(bb') = b'd(b) + b(db')$ (the Leibniz's rule).*

Remark that such a derivation is A-linear.

Definition 4.4 (Module of relative differential forms)
Let B be an A-algebra. A pair $\left(d, \Omega_{B/A}\right)$, where $d : B \longrightarrow \Omega_{B/A}$ is an A-derivation, is called the module of relative differential forms of B over A *if it satisfies the following universal property:*
for each A-derivation $d' : B \longrightarrow M$, there exists a unique morphism of B-modules $f : \Omega_{B/A} \longrightarrow M$ such that the following diagram commutes

$$\begin{tikzcd} B \arrow[r, "d'"] \arrow[d, "d"'] & M \\ \Omega_{B/A} \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

Example 4.5
Let K be a field and let B be the K-algebra $K[x]$. Then, we have $\Omega_{K[x]/K} \cong K[x]$. Indeed, we set $d(x) = 1$ and then we extend the map by K-linearity and set $d(x^n) = n \cdot d(x^{n-1})$ (for the Leibniz's rule). Now, if we have another K-derivation $d' : K[x] \longrightarrow M$, then we set $f(1) = d'(x)$ and extend the map by K-linearity and via the Leibniz's rule.

More generally, we have the following example.

Example 4.6
Let A be a ring and consider the A-algebra $B = A[x_1, \dots, x_n]$. Then, we have $\Omega_{B/A} \cong B^{(n)}$, the free B-module of rank n. We want to show that $B^{(n)}$ satisfies the universal property. First, we define

$$\begin{aligned} d : B &\longrightarrow B^n \\ b &\longmapsto \left(\frac{\partial b}{\partial x_i}\right), \end{aligned}$$

where $\frac{\partial b}{\partial x_i}$ denotes the formal derivative of b with respect to x_i. If $d' : B \longrightarrow M$ is another derivation, then we must have $d'(b) = \sum_{i=1}^{n} \frac{\partial b}{\partial x_i} d'(x_i)$ (by the Leibniz's rule and the A-linearity of d'). Therefore, we define $f_i : B \longrightarrow M$ by $f_i(\tilde{b}) = \tilde{b} \cdot d'(x_i)$. The morphism $f : B^{(n)} \longrightarrow M$ induced by the f_i is the required morphism.

Example 4.7
Let k be a field and K be a separable algebraic extension of k. Then, we have $\Omega_{K/k} = 0$. To see this equality, consider some $\alpha \in K$ and the polynomial $f \in k[t]$ such that $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Then, using induction, Leibniz's rule and k-linearity, we get $d(\alpha) \cdot f'(\alpha) = d\big(f(\alpha)\big) = 0$, which implies $d(\alpha) = 0$.

Proposition 4.8
Let B be an A-algebra. Then, the module of relative differential forms $\Omega_{B/A}$ exists.

Proof. Let F denote the free B-module generated by the symbols $\big\{d(b) : b \in B\big\}$ and let K be the sub-B-module of F generated by

$$\{a : a \in A\} \cup \{d(bb') - b'\,d(b) - b\,d(b') : b, b' \in B\} \cup \{d(b+b') - d(b) - d(b') : b, b' \in B\}.$$

We define $d : B \longrightarrow F/_K$, which maps any $b \in B$ to $\overline{d(b)}$. Then, one can check that the quotient $F/_K$ has the required properties. □

Proposition 4.9 (Functoriality of $\Omega_{-/A}$)
Let $\psi : B \longrightarrow C$ be a morphism of A-algebras. Then, there is a canonical morphism of B-algebras $\overline{\psi} : \Omega_{B/A} \longrightarrow \Omega_{C/A}$.

Proof. It is sufficient to note that $d \circ \psi : B \longrightarrow \Omega_{C/A}$ is an A-derivation of B. □

Corollary 4.10
We have a canonical morphism $\alpha : \Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A}$.

On the other hand, we have a morphism $\beta : \Omega_{C/A} \longrightarrow \Omega_{C/B}$.

Proposition 4.11
Let B and C be two A-algebras. The sequence

$$\Omega_{B/A} \otimes_B C \xrightarrow{\alpha} \Omega_{C/A} \xrightarrow{\beta} \Omega_{C/B} \longrightarrow 0$$

is exact.

Proof. See [Liu06, Proposition 6.1.8]. □

Proposition 4.12
Let A' and B be two A-algebras and let $B' = B \otimes_A A'$. Then, we have the isomorphism $\Omega_{B'/A} \cong \Omega_{B/A} \otimes_B B'$.

Proof. First, we note that $\Omega_{B/A} \otimes_B B' \cong \Omega_{B/A} \otimes_A A'$. We want to show that this B' module satisfies the universal property of the module of relative differential forms of B' over A'. First, we define

$$\begin{aligned} \varphi : B \times A' &\longrightarrow \Omega_{B/A} \otimes_A A' \\ (b, a') &\longmapsto d(b) \otimes a', \end{aligned}$$

where $d : B \longrightarrow \Omega_{B/A}$ is the A-derivation. This A-bilinear map induces a morphism $\tilde{\varphi}$ from $B' = B \otimes_A A'$ to $\Omega_{B/A} \otimes_A A'$. Moreover, it is easy to see that $\tilde{\varphi}$ satisfies the Leibniz rule and is A'-linear. Now, suppose we are given an A'-derivation $\psi : B' \longrightarrow M$. We have to show there exists a unique $\eta : \Omega_{B/A} \otimes_A A' \longrightarrow M$ such that $\eta \circ \tilde{\varphi} = \psi$. We define the following morphism:

$$\begin{aligned} \tilde{\psi} : \Omega_{B/A} \times A' &\longrightarrow M \\ (d(b), a') &\longmapsto \psi(b \otimes a'). \end{aligned}$$

One can check that it is well-defined and A-bilinear. Thus, we get a unique morphism η such that the following diagram commutes

$$\begin{tikzcd}
& B' \arrow[rr, "\psi"] \arrow[dd, "\tilde{\varphi}"] & & M \\
B \times A' \arrow[ur, "\tau"] \arrow[dr, "\varphi"'] & & & \\
& \Omega_{B/A} \otimes_A A' \arrow[uurr, dashed, "\eta"] & & \Omega_{B/A} \times A', \arrow[ll, "\tau"'] \arrow[uu, "\tilde{\psi}"']
\end{tikzcd}$$

as required. □

4.2.2 Global definition of the module of the relative differential forms

Recall 4.13
Let A be a ring and M be an A-module and let $X = \operatorname{Spec} A$. The *sheaf associated* to M on $\operatorname{Spec} A$, denoted by $\tilde{M}$, has the following properties:

(i) $\widetilde{M}$ is an $\mathcal{O}_X$-module;

(ii) for each $\mathfrak{p} \in \operatorname{Spec} A$, we have $\left(\widetilde{M}\right)_{\mathfrak{p}} \cong M_{\mathfrak{p}}$;

(iii) for any $f \in A$, the A_f-module $\tilde{M}\big(D(f)\big)$ is isomorphic to M_f;

(iv) $\widetilde{M}(X) \cong M$.

See [Har77, II.5] for the construction and the properties of $\tilde{M}$.
Now, if $\mathscr{F}$ is any $\mathcal{O}_X$-module, we say that $\mathscr{F}$ is *quasi-coherent* if there exists an affine covering $U_i = \operatorname{Spec} A_i$ of X such that for each i there exists an A_i-module M_i with $\mathscr{F}\big|_{U_i} \cong \widetilde{M_i}$. If each M_i is a *finitely generated* A_i-module, we say that $\mathscr{F}$ is *coherent*.

Now, we would like to generalize the construction of the module of relative forms: if $f : X \longrightarrow Y$ is a morphism of schemes, we would like to define an $\mathcal{O}_X$-module $\Omega_{X/Y}$ such that:

(i) if $U \subset X$ and $V \subset Y$ are two open affine subsets such that $f(U) \subset V$, then $\Omega_{X/Y}\big|_U \cong \left(\Omega_{\mathcal{O}_X(U)/\mathcal{O}_Y(V)}\right)^{\sim}$;

(ii) for each $x \in X$, then $\left(\Omega_{X/Y}\right)_x \cong \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}}$.

To do this, there exists two construction. The former reproduces the construction of the structural sheaf on an affine scheme $\operatorname{Spec} R$ (we associate to each open set a collection of functions satisfying certain properties), while the latter is more abstract. The first construction is presented in [Liu06, Proposition 6.1.17]. To introduce the second one, we present another definition of $\Omega_{B/A}$ for an A-algebra B.

Proposition 4.14 (An alternative definition of $\Omega_{B/A}$)
Let B be an A-algebra. Let $\Delta : B \otimes_A B \longrightarrow B$, which sends any $x \otimes y$ to xy, and denote by I the kernel of Δ. Then, $I/_{I^2} \cong \Omega_{B/A}$.

Proof. We want to define the morphisms ε, η, ψ and $\overline{\psi}$ (in this order) such that the following diagram commute:

$$\begin{tikzcd}
A \arrow[r, "d"] \arrow[d, "\varepsilon"'] & \Omega_{A/B} \arrow[dd, bend left, "\eta"] \\
I \arrow[ur, "\psi"] \arrow[d, "\pi"'] & \\
I/_{I^2}, \arrow[uur, bend left, "\overline{\psi}"] &
\end{tikzcd}$$

where $d : A \longrightarrow \Omega_{A/B}$ and $\pi : I \longrightarrow I/_{I^2}$ are the canonical maps. Since we want $\pi \circ \varepsilon$ to be an B-derivation, we define $\varepsilon(b) = 1 \otimes b - b \otimes 1$. Then, it is easily seen that $\varepsilon(b) \in \ker \Delta$ and that $\pi\varepsilon$ is an A-derivation. Therefore, we get an induced map $\eta : \Omega_{B/A} \longrightarrow I/_{I^2}$. Now, if $\sum x_i \otimes y_i \in I$, we have $\sum x_i y_i = 0$. Thus, we can write

$$\sum x_i \otimes y_i = \sum (x_i \otimes y_i - x_i y_i \otimes 1) = \sum x_i\, \varepsilon(y_i).$$

Hence, we define $\psi\big(\sum x_i \otimes y_i\big) := \sum x_i\, d(y_i)$ and one can check that ψ passes to the quotient. Therefore, we get $\overline{\psi} : I/_{I^2} \longrightarrow \Omega_{B/A}$. Finally, η and $\overline{\psi}$ are inverses of each other. ☐

Before giving the definition, we recall a few results.

Definition 4.15 (Ideal sheaf of a closed immersion)
Let $f : X \longrightarrow Y$ be a closed immersion. Then, we define the ideal sheaf *of X, denoted $\mathcal{J}_{X/Y}$, as the kernel of the morphism $i^\sharp : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$.*

Proposition 4.16
Let Y be a scheme and X be a closed subscheme of Y. Then, $\mathcal{J}_{X/Y}$ is a quasi-coherent sheaf of ideals on Y.

Proof. See [Har77, Proposition II.5.9]. ☐

Proposition 4.17
Let X and Y be schemes and let $f : X \longrightarrow Y$ be a morphism of schemes. Then the diagonal morphism $\Delta : X \longrightarrow X \times_Y X$ induces an isomorphism from X onto $\Delta(X)$ which is a locally closed subscheme of Y (i.e., a closed subscheme of an open subscheme W of Y).

Proof. See [Har77, Corollary II.4.2]. ☐

Definition 4.18 (Sheaf of relative differential forms)
Let $f : X \longrightarrow Y$ be a morphism of schemes and let W be as in the previous proposition: this means that we have the following factorization of Δ

$$\begin{tikzcd} X \arrow[rr, "\Delta"] \arrow[dr, "g"'] & & X \times_Y X \\ & W \arrow[ur, hook] & \end{tikzcd}$$

Let $\mathcal{J}_{X/Y}$ be the ideal sheaf of g on W. Then, we define the sheaf of relative differential forms, *denoted $\Omega_{X/Y}$, as $g^*\left(\mathcal{J}_{X/Y}/\mathcal{J}^2_{X/Y}\right)$.*

Remark 4.19
Let $\varphi : A \longrightarrow B$ be a morphism of rings and $f : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ be the corresponding morphism. Then, we have $\Omega_{\operatorname{Spec} B/\operatorname{Spec} A} \cong \left(\Omega_{B/A}\right)^\sim$.

Example 4.20
We know that if $\varphi : A \longrightarrow B$ is surjective, then $\Omega_{B/A} = 0$. In particular, if $\varphi : A \longrightarrow B = A/_I$ is the canonical map, then $\Omega_{B/A}$. This implies that if $Y = \operatorname{Spec} R$ and $f : X \longrightarrow Y$ is a closed immersion, then $\Omega_{X/Y}$ is trivial.

Example 4.21
If $X = \mathbb{A}^n_K$, then we have $\Omega_{X/k} = \mathcal{O}_X^{(n)}$, the free $\mathcal{O}_X$-module of rank n (see Example 4.6).

4.3 Étale morphisms

In this section, we follow [Mil80]. In particular, every scheme is assumed to be locally noetherian[1].

Definition 4.22 (Unramified morphism)
Let $f : X \longrightarrow Y$ be a morphism locally of finite type. Let $x \in X$ and $y = f(x)$. We say that f is unramified *at x if $\mathcal{O}_{X,x}/_{\mathfrak{m}_y\mathcal{O}_{X,x}}$ is a finite separable extension of $\kappa(y)$, where $\mathfrak{m}_y\mathcal{O}_{X,x}$ denote the ideal generated by the image of $\mathfrak{m}_y$ under the ring homomorphism $f^\sharp_x : \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$. We say that f is* unramified *if it is unramified at every point of X.*

Remark 4.23
The condition "$\mathcal{O}_{X,x}/_{\mathfrak{m}_y\mathcal{O}_{X,x}}$ is a finite separable extension of $\kappa(x)$" is equivalent to the two conditions:

(i) $\kappa(x)$ is a finite and separable extension of $\kappa(y)$;

(ii) $\mathfrak{m}_y\mathcal{O}_{X,x} = \mathfrak{m}_x$.

Proposition 4.24
Let $f : X \longrightarrow Y$ be a morphism of schemes of locally finite type. Let $x \in X$, $y = f(x)$ and the corresponding point $x' \in X_y$. Then, f is unramified at x if and only if $f' : X_y \longrightarrow \kappa(y)$, the morphism given by the universal property of the fibred product, is unramified at x'.

Proof. Follows from Proposition 1.15. $\square$

Definition 4.25 (Étale morphism)
Let $f : X \longrightarrow Y$ be a morphism locally of finite type and let $x \in X$. We say that f is étale *at x if f is flat and unramified at x. We say that f is* étale *if it is étale at every point of X.*

[1] Recall that a scheme is locally noetherian if it can be covered by the spectras of noetherian ring. Moreover, a scheme X is locally noetherian if and only if for every open affine subset $\operatorname{Spec} R \cong U$ of X, R is a noetherian ring (see Proposition [Har77, II.3.2]).

Example 4.26
The morphism $\operatorname{Spec} F \longrightarrow \operatorname{Spec} K$ corresponding to a finite field extension $K \subset F$ is unramified (and hence étale) if and only if F is separable over K.

Example 4.27
A closed immersion, which corresponds locally to a homomorphism $R \longrightarrow R/_{I}$, is unramified.

Example 4.28
Let K be a field, $Q \in K[t]$ be a monic polynomial and consider the morphism $f : X = \operatorname{Spec} K[t]/_{\langle Q\rangle} \longrightarrow \operatorname{Spec} K$. Let $x \in X$, which corresponds to the ideal $\langle P\rangle/_{\langle Q\rangle}$, where P is an irreducible factor of Q, and denote by y the corresponding point of $\operatorname{Spec} K$. Then, f is unramified at x if and only if P is separable and $P^k \nmid Q$ if $k > 1$. In particular, f is unramified if and only if Q is separable. For the first condition, we note that $\kappa(x) \cong K[x]/_{\langle P\rangle}$. Hence, $\kappa(x)$ is a finite separable extension of $\kappa(y)$ if and only if P is separable. Now, suppose that $Q = P \cdot S$, with $P \nmid S$. Since $\mathfrak{m}_y \mathcal{O}_{X,x} = 0$, we have to show that $\mathfrak{m}_x = 0$. But an element in $\mathfrak{m}_x$ is of the form $\frac{g \cdot P + \langle Q\rangle}{h + \langle Q\rangle}$, for some $g, h \in K[t]$ with $h \notin \langle P\rangle$. Since $P \nmid S$, we have

$$\frac{g \cdot P + \langle Q\rangle}{h + \langle Q\rangle} = \frac{g \cdot P \cdot S + \langle Q\rangle}{h \cdot S + \langle Q\rangle} = 0,$$

and thus $\mathfrak{m}_x = 0$. Hence, f is unramified at x if P is separable and is a simple factor of Q. Now, suppose that $Q = P^k \cdot S$ with $P \nmid S$ and $k > 1$. In the ring $\big(K[x]/_{\langle Q\rangle}\big)_{\langle P\rangle/\langle Q\rangle}$, the element P is not zero and is not invertible. It follows that $\mathfrak{m}_x \neq 0$ and thus $\mathfrak{m}_y \mathcal{O}_{X,x} \neq \mathfrak{m}_x$. Therefore, f is not unramified at x.
Since $K[t]/_{\langle Q\rangle}$ is a free K-module of rank equal to the degree of Q, the morphism $f : K[t]/_{\langle Q\rangle} \longrightarrow \operatorname{Spec} K$ is flat. Hence, it is étale if and only if Q is separable.

Example 4.29
Let K be a field of characteristic different from 2 and consider the projection of the parabola $f : X = \operatorname{Spec} K[x,y]/_{\langle y^2 - x\rangle} \longrightarrow Y = \operatorname{Spec} K[x]$. Let a be a closed point of Y. Then, we find $X_a = \operatorname{Spec}\Big(K[y]/_{\langle y^2 - a\rangle}\Big)$ (see Example 1.12). Now, we have three cases:

$\mathbf{a = 0}$ We have $X_a = \operatorname{Spec}\Big(K[y]/_{\langle y^2\rangle}\Big)$ and the morphism $X_0 \longrightarrow K$ is ramified.

$\mathbf{a = b^2}$ We have $X_a = \operatorname{Spec}\big(K \times K\big)$ and the morphism $X_{b^2} \longrightarrow K$ is unramified.

a is not a square We have $X_a = \operatorname{Spec} F$, where F is a separable extension of degree two of K. Hence, $X_a \longrightarrow K$ is unramified.

Using the previous proposition we see that f is unramified at (a, b) provided that $(a, b) \neq 0$.

Example 4.30
Let $f : X = \operatorname{Spec} K[x,y]/_{\langle x^2 - y\rangle} \longrightarrow Y = \operatorname{Spec} K[x]$ be the projection of the parabola on the x-axis. If a is an element of K, we consider the morphism $f_a : X_a \longrightarrow \operatorname{Spec} \kappa(a)$ (given by the universal property of the fibred product) which is, up to isomorphism, $f_a : \operatorname{Spec} K \longrightarrow \operatorname{Spec} K$. Hence, f is unramified at a (see Proposition 4.24). Now, on the fiber over the generic point $\langle 0\rangle$ of Y, we get the morphism $\operatorname{Spec} K[x] \longrightarrow \operatorname{Spec} K[x]$. Hence, f is unramified.

Example 4.31
Let d be a square-free integer and consider the number field $K = \mathbb{Q}[\sqrt{d}]$ and

the ring of integers $\mathcal{O}_K$ of K, that is

$$\mathcal{O}_K = \begin{cases} \mathbb{Z}[\sqrt{d}] & \text{if } d \equiv 2,3 \pmod 4, \\ \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right] & \text{if } d \equiv 1 \pmod 4. \end{cases}$$

We consider the morphism $f : X = \operatorname{Spec} \mathcal{O}_K \longrightarrow \operatorname{Spec} \mathbb{Z}$ and want to determine at which points f is unramified. Let $p \in \mathbb{P}$ and suppose $d \equiv 2,3 \pmod 4$. Then, we have

$$X_p = \mathbb{Z}[\sqrt{d}] \otimes_{\mathbb{Z}} \mathbb{F}_p \cong \mathbb{Z}[x]/\langle x^2 - d\rangle \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \cong \mathbb{F}_p[x]/\langle x^2 - \overline{d}\,\rangle.$$

(i) If $p \mid d$, then we have $\mathbb{F}_p[x]/\langle x^2\rangle$, which is ramified over $\mathbb{F}_p$ (see Example 4.28).

(ii) If $p \nmid d$ and d is a square mod p, then $x^2 - \overline{d}$ is separable in $\mathbb{F}_p[x]$ and so $\mathbb{F}_p[x]/\langle x^2 - \overline{d}\,\rangle$ is unramified over $\mathbb{F}_p$

(iii) If d is not a square mod p, then $x^2 - \overline{d}$ is irreducible and it is separable if $p \neq 2$.

Hence, if $p \mid \operatorname{disc}(K) = 4d$, then f is ramified at p. If $p \nmid 4d$, then it depends on the Legendre symbol and on if $p = 2$ or not.

4.3.1 First properties of étale morphisms and other characterizations

We recall the three following definitions.

Definition 4.32 (Jacobson ideal)
The Jacobson radical of a ring is the intersection of all its maximal ideals.

Definition 4.33 (Separable algebra)
Let K be a field and A a K-algebra. We say that A is separable *(over K) if the Jacobson radical of $\overline{A} = A \otimes_K \overline{K}$ is zero, where $\overline{K}$ is the algebraic closure of K.*

Definition 4.34 (Separably closed field)
We say that a field K is separably closed *if every separable element of $\overline{K}$ belongs to K.*

Proposition 4.35
Let $f : X \longrightarrow Y$ be a morphism of locally finite type. The followings are equivalent:

(i) f is unramified.

(ii) For all $y \in Y$, the morphism $X_y \longrightarrow \kappa(y)$ is unramified.

(iii) For every morphism $\operatorname{Spec} K \longrightarrow Y$, with K separably closed, the morphism $X \times_Y \operatorname{Spec} K \longrightarrow \operatorname{Spec} K$ is also unramified (all the geometric fibers of f are unramified).

(iv) For every $y \in Y$, X_y has an open covering by spectra of finite separable $\kappa(x)$-algebras.

(v) For every $y \in Y$, X_y is an amalgamated sum $\coprod \operatorname{Spec} K_i$, where the K_i are finite separable extensions of $\kappa(x)$.

Proof. See 4.24 for $(i) \Leftrightarrow (ii)$ and [Mil80, Proposition 3.2]. □

Proposition 4.36
Let $f : X \longrightarrow Y$ be a morphism of locally of finite type. Then, the followings are equivalent:

(*i*) *f is unramified;*

(*ii*) $\Omega_{X/Y} = 0$*;*

(*iii*) *the diagonal morphism $\Delta_{X/Y} : X \longrightarrow X \times_Y X$ is an open immersion.*

Proof. (*i*) ⇒ (*ii*) Let $x \in X$ and $y = f(x)$. We know that $\Omega_{\kappa(x)/\kappa(y)} = 0$ (see Example 4.7). Since $\kappa(x) = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \kappa(y)$, Proposition 4.12 implies

$$0 = \Omega_{\kappa(x)/\kappa(y)} \cong \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,y}} \cong \mathcal{O}_{X,x}\kappa(x).$$

Since f is locally of finite type, then $\Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,y}}$ is a finitely generated $\mathcal{O}_{X,x}$-module and Corollary A.6 implies $0 = \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,y}} \cong \left(\Omega_{X/Y}\right)_x$. Hence, $\Omega_{X/Y} = 0$, as required.

(*ii*) ⇒ (*iii*) Let W and $\mathcal{J}_{X/Y}$ as in Definition 4.18. By hypothesis, we have the equality $\left(\mathcal{J}_{X/Y}\right)_x / \left(\mathcal{J}_{X/Y}\right)_x^2 = 0$ and Corollary A.7 implies that $\left(\mathcal{J}_{X/Y}\right)_x = 0$ for every $x \in X$. Hence, $\left(\mathcal{J}_{X/Y}\right)$ is zero on an open subset V of U containing X. Therefore, we have $(X, \mathcal{O}_X) \cong \left(V, \mathcal{O}_{X\times_Y X}\big|_U\right)$, as required.

(*iii*) ⇒ (*i*) First, suppose that $f : X \longrightarrow \operatorname{Spec} K$, where K is some algebraically closed field. Let $x \in X$ be some closed point of x. We have the inclusion $K \longrightarrow \kappa(x)$ which means that $K = \kappa(x)$, since K is algebraically closed. Hence, we get a section $g : \operatorname{Spec} K \longrightarrow X$ of f whose image is $\{x\}$. Now, we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\Delta"] & X \times_Y X \\ \{x\} \arrow[u, "g"] \arrow[r, "g"'] & X. \arrow[u, "(g\circ f, \mathrm{id})"'] \end{tikzcd}$$

Now, since Δ is an open immersion, $\{x\}$ is open in X. Furthermore, the morphism $\{x\} = \operatorname{Spec} \mathcal{O}_{X,x} \longrightarrow \operatorname{Spec} K$ satisfies the property that the induced morphism $\operatorname{Spec} \mathcal{O}_{X,x} \longrightarrow \operatorname{Spec}\left(\mathcal{O}_{X,x} \otimes_K \mathcal{O}_{X,x}\right)$ is still an open immersion. Since $\mathcal{O}_{X,x}$ is an artinian ring with residue field K, the ring $\mathcal{O}_{X,x} \otimes_K \mathcal{O}_{X,x}$ has only one prime ideal and $\mathcal{O}_{X,x} \otimes_K \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}$ must be an isomorphism. Considering the dimensions over K gives $\mathcal{O}_{X,x} = k$. Hence, $\mathfrak{m}_x = 0$ and f is unramified at x. Finally, Proposition 4.35 gives the required result.

□

Remark 4.37
In the last proof, we use for the first time the hypothesis of local finiteness.

Proposition 4.38
Let $f : X \longrightarrow Y$ be an étale morphism and let $x \in X$ and $y = f(x)$. Then, we have $\mathfrak{m}_y/\mathfrak{m}_y^2 \otimes_{\kappa(y)} \kappa(x) \cong \mathfrak{m}_x/\mathfrak{m}_x^2$.

Proof. We compute

$$\begin{aligned} \mathfrak{m}_y/\mathfrak{m}_y^2 \otimes_{\kappa(y)} \kappa(x) &\cong \mathfrak{m}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}/\mathfrak{m}_x \cong \left(\mathfrak{m}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}\right) \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}_x \\ &\cong \left(\mathfrak{m}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}\right) / \left(\mathfrak{m}_x \left(\mathfrak{m}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}\right)\right) \end{aligned}$$

Since f is flat at x, we have $\mathfrak{m}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \cong \mathfrak{m}_y \mathcal{O}_{X,x}$. Finally, since f is unramified at x, the last term is isomorphic to $\mathfrak{m}_x/\mathfrak{m}_x^2$. □

Proposition 4.39
Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two morphisms locally of finite type. Then:

- *If f is an open immersion, then f is étale.*
- *If f and g are étale, then so is $g \circ f$.*
- *Any base change of an étale morphism is étale.*

Proof. By Proposition 2.20, we only need to prove "unramified part" of the statements. Let $x \in X$, $y = f(x)$ and $z = g(y)$.

(i) Obvious.

(ii) If $\kappa(x)$ is separable over $\kappa(y)$ and $\kappa(y)$ is separable over $\kappa(z)$, then $\kappa(x)$ is separable over $\kappa(z)$. The second part is obvious.

(iii) Let $g : Y' \longrightarrow Y$ be some morphism of schemes. By Proposition 4.35, to show that $f' : X \times_Y Y' \longrightarrow Y'$ is unramified is equivalent to show that for each separably closed field K and each morphism $\operatorname{Spec} K \longrightarrow Y'$, the morphism $X \times_Y Y' \times_{Y'} \operatorname{Spec} K \longrightarrow \operatorname{Spec} K$ is unramified. But since we have the isomorphism $X \times_Y Y' \times_{Y'} \operatorname{Spec} K \cong X \times_Y \operatorname{Spec} K$, this is the case by the assumptions on f.

□

Proposition 4.40 (Jacobian criterion)
Let A be a noetherian ring and let $f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$ be polynomials. Then, $\operatorname{Spec} A[x_1, \ldots, x_n]/\langle f_1, \ldots, f_n \rangle$ *is étale over* $\operatorname{Spec} A$ *if and only if the determinant of the Jacobian matrix $\frac{\partial f_i}{\partial x_j}$ is a unit in $A[x_1, \ldots, x_n]/\langle f_1, \ldots, f_n \rangle$.*

Proof. See [Mil80, Example I.3.4]. □

The Jacobian criterion is really useful to determine if a morphism is étale or not.

Example 4.41
The morphism $\operatorname{Spec} \mathbb{Q}[x, y]/\langle y^2 - x \rangle \longrightarrow \operatorname{Spec} K[x]$ is not étale since y is not a unit in $\mathbb{Q}[x, y]/\langle y^2 - x \rangle$. The morphism $\operatorname{Spec} \mathbb{Q}[y, x]/\langle x^2 - y \rangle \longrightarrow \operatorname{Spec} K[x]$ is étale.

4.4 Étale topology

Proposition 4.42
Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be morphisms of schemes. Suppose that $g \circ f$ is étale and g is unramified. Then, f is étale.

Proof. See [Mil80, Corollary I.3.6]. □

Corollary 4.43
For a scheme X, $\mathfrak{Sch}_{\text{ét}}/X$ is a full subcategory of $\mathfrak{Sch}/X$.

Definition 4.44 (Zariski site)
The (small) étale site, *denoted $X_{ét}$, is $(\mathfrak{Sch}_{\text{ét}}/X)_{ét}$. The* (small) Zariski site, *denoted X_{zar}, is $(\mathfrak{Sch}_{\text{oi}}/X)_{zar}$.*

Proposition 4.45
Let X be a scheme. Then, $X_{ét}$ is a site.

Proof. We check the three conditions:

(i) We know that all isomorphisms are étale.

(ii) Let $\left\{ U_i \xrightarrow{\varphi_i} U \right\}_i$ be an étale covering and let $\varphi : V \longrightarrow U$ be a morphism. We know (Proposition 4.39) that the morphisms $V \times_U U_i \longrightarrow V$ is étale for every i. To show that these morphisms form a covering, consider $v \in V$ and let $u = \varphi(v)$. Now, pick $i \in I$ such that there exists some $u' \in U_i$ with $\varphi_i(u') = u$. The corollary 1.17 implies that there exists $v' \in V \times_U U_i$ which is mapped to v under the morphism $V \times_U U_i \longrightarrow V$, as required.

(iii) Comes from Proposition 4.39.

□

Remarks 4.46 (Presheaves and sheaves on $X_{\text{ét}}$)
Let F be a presheaf on $X_{\text{ét}}$.

(i) Unlike with the Zariski topology, it might exists many morphisms from U to X. In this case, there will be many restriction maps from $F(X)$ to $F(U)$.

(ii) We recall the condition for F to be a sheaf: for every covering $\left\{ U_i \xrightarrow{\varphi_i} U \right\}_i$ the following diagram (see Remark 3.19 for the definition of η and ψ) is an equalizer

$$F(U) \longrightarrow \prod_i F(U_i) \overset{\eta}{\underset{\psi}{\rightrightarrows}} \prod_{i,j} F\big(U_i \times_U U_j\big).$$

Since the φ_i are not necessarily monomorphisms, we may not have the isomorphism $U_i \times_U U_i \cong U_i$. Hence, the case where $i = j$ is not trivial (again, unlike in the Zariski case).

(iii) Let F be a sheaf. Since the empty set of morphisms form a covering of the empty set, the condition says that we have an exact sequence

$$F(\emptyset) \longrightarrow 0 \rightrightarrows 0$$

The universal property of the equalizer implies that $F(\emptyset)$ is a terminal object in $\mathscr{C}$. Hence, we have $F(\emptyset) = 0$. We note that this result holds also for sheaves over X_{zar}.

4.5 The case of $\operatorname{Spec} k$

In this section, we state without proof a theorem explaining the relationships between étale coverings theory and Galois theory. The details can be found in [Tam94]. Let k be a field and k_s be a separable closure. Let G be the Galois group of the Galois extension k_s/k (with its usual structure of profinite group).

Recall 4.47
Let X be a scheme and k be any field. We denote by $X(k)$ the set of k-points of X, which is the set of morphisms $\operatorname{Spec} k \longrightarrow X$. We know that each k-point of X corresponds uniquely to a point $x \in X$ and a homomorphism of fields $\kappa(x) \longrightarrow k$.

The group G acts on the left of $X(k_s)$ as follows:

$$\begin{aligned} G \times X(k_s) &\longrightarrow X(k_s) \\ (g, \sigma) &\longmapsto g \cdot \sigma = \sigma \circ \operatorname{Spec}(g), \end{aligned}$$

where $\operatorname{Spec}(g) : \operatorname{Spec}(k_s) \longrightarrow \operatorname{Spec}(k_s)$ is the image by the functor Spec of g. Let H be an open subgroup of G and let $k' = k^H$, the fixed field of H. Then, we can identify $X(k_s)^H$ with $X(k')$. Since H is a closed subgroup of G (recall that in a compact topological group, any open subgroup is closed), k' is a finite extension of k. The inclusion $k' \longrightarrow k_s$ induces a morphism $\operatorname{Spec} k_s \longrightarrow \operatorname{Spec} k'$ which induce the inclusion $X(k') \subset X(k_s)$. Moreover, since $X(k_s) = \bigcup_H X(k_s)^H$, as H goes through the set of open subgroups of G, G acts continuously of $X(k_s)$.

Before giving the main result, we recall that a topological group G gives rise to a site T_G (see Example 3.17).

Theorem 4.48
The functor which send an $(\operatorname{Spec} k)$-scheme X' to $X'(k_s)$ is an equivalence of topologies between the étale site $(\operatorname{Spec} k)_{\acute{e}t}$ of $\operatorname{Spec} k$ and the site T_G.

Proof. See [Tam94]. $\square$

5 Fpqc topology and the descent problem

In this section, we follow the paper [Bro11]. Additional information and proofs can be found in [G$^+$, Exposés VIII].

5.1 Some examples for the descent problem

Before giving the general setting of the descent problem, we give two examples: gluing morphisms and gluing schemes.

5.1.1 Gluing morphisms for the Zariski topology

Let U be a scheme and let X and Y two U-schemes. If $\{U_i\}_i$ is a covering of U for the Zariski topology, we let X_i and Y_i the preimages of the U_i under the φ_i, that is $X_i = X \times_U U_i$ and $Y_i = Y \times_U U_i$. Now, suppose we are given a family of morphisms $\varphi_i : X_i \longrightarrow Y_i$ which agree on the intersection $X_i \cap X_j$ (recall that $X_i \cap X_j \cong X_i \times_X X_j$). We know that there exists a unique morphisms of schemes $\varphi : X \longrightarrow Y$ such that $\varphi\big|_{X_i} = \varphi_i$. We can rephrase this result as follows:

Proposition 5.1
Let X be a schemes and let X and Y be two S-schemes. Consider the following functor

$$\begin{aligned} F : \mathfrak{Sch}(S) &\longrightarrow \mathbf{Ens} \\ U &\longmapsto \mathrm{Hom}_{\mathfrak{Sch}(U)}\big(X \times_S U, Y \times_S U\big). \end{aligned}$$

Then, F is a sheaf for the Zariski topology.

5.1.2 Relative gluing schemes for the Zariski topology

Let U be a scheme and let $\{U_i\}_i$ be a Zariski covering of U. As usual, we denote by U_{ij} and U_{ijk} the sets $U_i \cap U_j$ and $U_i \cap U_j \cap U_k$ for every i, j, k. For each i, we consider some U_i-scheme $f_i : X_i \longrightarrow U_i$. The goal here is to glue the X_i to get an U-scheme X. Suppose that we have the followings:

(i) For all i, j, we have an isomorphism $\varphi_{ij} : f_j^{-1}(U_{ij}) \longrightarrow f_i^{-1}(U_{ij})$, where $f_j^{-1}(U_{ij})$ denotes the pullback of the scheme U_{ij} under f_j.

(ii) For all i, j, k, we have the cocycle condition

$$\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$$

which means that the following diagram is commutative:

$$\begin{tikzcd} f_k^{-1}(U_{ijk}) \arrow[rr, "\varphi_{jk}"] \arrow[dr, "\varphi_{ik}"'] & & f_j^{-1}(U_{ijk}) \arrow[dl, "\varphi_{ij}"] \\ & f_i^{-1}(U_{ijk}). & \end{tikzcd}$$

Then, there exists a unique U-scheme $f : X \longrightarrow U$ and isomorphisms $\varphi_i : f^{-1}(U_i) \longrightarrow X_i$ which make the following diagram commutative

$$\begin{tikzcd} f_j^{-1}(U_{ij}) \arrow[rr, "\varphi_j^{-1}"] \arrow[dr, "\varphi_{ij}"'] & & f^{-1}(U_{ij}) \arrow[dl, "\varphi_i"] \\ & f_i^{-1}(U_{ij}). & \end{tikzcd}$$

5.2 The fpqc site

Definition 5.2 (Quasi-compact)
A morphism of schemes $f : X \longrightarrow Y$ is quasi-compact *if there exists an open covering of affine subsets V_i of Y such that $f^{-1}(V_i)$ is compact (recall that we do not suppose any separateness or Hausdorff condition) for each i.*

Definition 5.3 (Fpqc morphism)
A morphism of schemes $f : X \longrightarrow Y$ is fpqc *if it is faithfully flat (see Definition 2.17) and quasi-compact*[2]*.*

Proposition 5.4
The composition of two quasi-compact morphisms is quasi-compact.

Proof. See [GD71, I.6]. □

Proposition 5.5
The base change of a quasi-compact morphism is quasi-compact.

Proof. See [GD71, I.6]. □

The last proposition, Corollary 1.18 and Proposition 2.20 now imply.

Proposition 5.6
The base change of a fpqc morphism is again fpqc.

Definition 5.7 (The fpqc site)
Let S be a scheme. The fpqc site *on S is the category $\mathfrak{Sch}(S)$ together with the following of coverings: a collection $\left\{ U_i \xrightarrow{\varphi_i} U \right\}$ is a covering of the S-scheme U if the induced morphism $\coprod_i U_i \longrightarrow U$ is fpqc.*

Proposition 5.8
The fpqc site is indeed a site.

Proof. (i) It is clear that isomorphisms are fpqc coverings.

(ii) Let $\left\{ U_i \longrightarrow U \right\}$ be a fpqc covering and $V \longrightarrow U$. Since $\left\{ U_i \longrightarrow U \right\}$ is an fpqc covering and since the property of being fpqc is stable under base change, the morphism $\left(\coprod_i U_i \right) \times_U V \longrightarrow V$ is also fpqc. Therefore, the morphism $\coprod_i \left(U_i \times_U V \right) \longrightarrow V$ is also fpqc, as required.

(iii) Let $\left\{ U_i \xrightarrow{\varphi_i} U \right\}_i \in \mathrm{Cov}(\mathscr{T})$ be a covering and let $\left\{ V_{ij} \xrightarrow{\psi_{ij}} U_i \right\}_j \in \mathrm{Cov}(\mathscr{T})$ be coverings for every i. We have show that the morphism $\coprod_{i,j} V_{ij} \longrightarrow U$ is fpqc. But this morphism factors through

$$\coprod_{i,j} V_{i,j} \longrightarrow \coprod_i U_i \longrightarrow U.$$

Now, if a family of morphisms $X_i \longrightarrow X$ is quasi-compact, then so is the morphism $\coprod_i X_i \longrightarrow X$ and the same holds for the flatness and the surjectivity. Hence, our morphism is fpqc, as required. □

[2] The term fpqc comes from the French "fidèlement plat et quasi-compact".

5.3 The descent problem

Definition 5.9 (Contravariant pseudo-functor)
Let $\mathscr{C}$ be a category. A contravariant pseudo-functor *$\mathscr{F}$ on $\mathscr{C}$ consists of the followings:*

(i) *For every object U of $\mathscr{C}$, a category $\mathscr{F}(U)$.*

(ii) *For every morphism $f : U \longrightarrow V$ in $\mathscr{C}$, a functor $f^* : \mathscr{F}(V) \longrightarrow \mathscr{F}(U)$.*

(iii) *For every object U in $\mathscr{C}$, an isomorphism (of functors) $\varepsilon_U : (\mathrm{id}_U)^* \longrightarrow \mathrm{id}_{\mathscr{F}(U)}$.*

(iv) *For every pair of morphisms $U \xrightarrow{f} V \xrightarrow{g} W$, an isomorphism (of functors) $\alpha_{f,g} : f^* \circ g^* \cong (g \circ f)^* : \mathscr{F}(W) \longrightarrow \mathscr{F}(U)$.*

Moreover, $\mathscr{F}$ have to satisfy the following "compatibility conditions":

(i) *For every morphism $f : U \longrightarrow V$ in $\mathscr{C}$ and every object η in $\mathscr{F}(V)$, we have*

$$\begin{aligned}\left(\alpha_{\mathrm{id}_U,f}\right)_\eta &= \left(f^* \circ \varepsilon_V\right)_\eta : f^* \circ (\mathrm{id}_V)^*(\eta) \longrightarrow f^*(\eta)\\ \left(\alpha_{f,\mathrm{id}_U}\right)_\eta &= \left(f^* \circ \varepsilon_V\right)_\eta : f^* \circ (\mathrm{id}_V)^*(\eta) \longrightarrow f^*(\eta).\end{aligned}$$

(ii) *For every triplet of morphisms $U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} T$ in $\mathscr{C}$ and every object $\eta \in \mathscr{F}(T)$, we have a commutative diagram:*

$$\begin{tikzcd} f^* \circ g^* \circ h^*(\eta) \arrow[r, "\left(\alpha_{f,g}\right)_{h^*(\eta)}"] \arrow[d, "f^*\left(\left(\alpha_{g,h}\right)_\eta\right)"'] & (g \circ f)^* \circ h^*(\eta) \arrow[d, "\left(\alpha_{g\circ f,h}\right)_\eta"] \\ f^* \circ (h \circ g)^*(\eta) \arrow[r, "\left(\alpha_{f,h\circ g}\right)_\eta"'] & (h \circ g \circ f)^*(\eta) \end{tikzcd}$$

Example 5.10
Let S be a scheme and let $\mathscr{C}$ be the category of S-schemes. To every S-scheme U, we associate the category $\mathfrak{Coh}(U)$ of quasi-coherent modules over U. Now, if $f : U \longrightarrow V$ is a morphism of S-scheme, $f^* : \mathfrak{Coh}(V) \longrightarrow \mathfrak{Coh}(U)$ is the usual pullback: a quasi-coherent module $\mathscr{M}$ over V is mapped to $f^{-1}(\mathscr{M}) \otimes_{f^{-1}\mathcal{O}_V} \mathcal{O}_U$.

Example 5.11
As before, let $\mathscr{C}$ denote the category of S-scheme. To any S-scheme U we associate the category of U-schemes. Now, if $f : U \longrightarrow V$ is any morphism of S-scheme, a V-scheme X is sent to $U \times_V X$. If $g : X \longrightarrow X'$ is a morphism of V-schemes, then $f^*(g) : U \times_V X \longrightarrow U \times_V X'$ is the base change of g.

The next definition generalizes the relative gluing of schemes for the Zariski topology.

Definition 5.12 (Descent data)
Let $\mathscr{C}$ be a site and $\mathscr{F}$ a pseudo-functor on $\mathscr{C}$. Let U be an object of $\mathscr{C}$ and let $\mathcal{U} = \left\{U_i \longrightarrow U\right\}$ a covering for U. For i, j, k, we denote by U_{ij} the fibred product $U_i \times_U U_j$ and by U_{ijk} the product $U_i \times_U U_j \times_U U_k$. Now, we have the canonical projections:

$$\begin{aligned}&\mathrm{pr}_1 : U_{ij} \longrightarrow U_i, \quad \mathrm{pr}_2 : U_{ij} \longrightarrow U_j,\\ &q_1 : U_{ijk} \longrightarrow U_i, \quad q_2 : U_{ijk} \longrightarrow U_j, \quad q_3 : U_{ijk} \longrightarrow U_k\\ &\mathrm{pr}_{12} : U_{ijk} \longrightarrow U_{ij}, \quad \mathrm{pr}_{23} : U_{ijk} \longrightarrow U_{jk}, \quad \mathrm{pr}_{13} : U_{ijk} \longrightarrow U_{ik}.\end{aligned}$$

For every i, let ξ_i be an object of $\mathscr{F}(U_i)$. A descent data *for the family $\{\xi_i\}$ is a collection of isomorphisms*

$$\varphi_{ij} : \mathrm{pr}_2^* \xi_j \xrightarrow{\cong} \mathrm{pr}_1^* \xi_i$$

in the category $\mathscr{F}(U_{ij})$. Since we have

$$\mathrm{pr}_{13}^* \circ \mathrm{pr}_2^* \cong \left(\mathrm{pr}_2 \circ \mathrm{pr}_{13} \right)^* = q_3^* : \mathscr{F}(U_{ijk}) \longrightarrow \mathscr{F}(U_k),$$

and similarly for other indices, we ask that we have

$$\mathrm{pr}_{13}^* \left(\varphi_{ik}\right) = \mathrm{pr}_{12}^* \left(\varphi_{ij}\right) \circ \mathrm{pr}_{23}^* \left(\varphi_{jk}\right) : q_3^*(\xi_k) \longrightarrow q_1^*(\xi_i),$$

up to the isomorphisms, for every triplet (i, j, k). Note that this is the generalization of the cocycle condition of Section 5.1.2. We say that $\left(\{\xi_i\}, \{\varphi_{ij}\}\right)$ is an object with a descent data for the covering $\mathcal{U} = \left\{ U_i \longrightarrow U \right\}$.

Definition 5.13 (The category of objects with descent data with respect to a covering)
Let $\left(\{\xi_i\}, \{\varphi_{ij}\}\right)$ and $\left(\{\eta_i\}, \{\psi_{ij}\}\right)$ be two objects with descent data for a covering $\mathcal{U} = \left\{ U_i \longrightarrow U \right\}$. A morphism from $\left(\{\xi_i\}, \{\varphi_{ij}\}\right)$ to $\left(\{\eta_i\}, \{\psi_{ij}\}\right)$ is a collection of morphisms $\alpha_i : \xi_i \longrightarrow \eta_i$ in $\mathscr{F}(U_i)$ such that the following diagram commute for every i, j:

$$\begin{tikzcd}
\mathrm{pr}_2^* \xi_j \arrow[r, "\mathrm{pr}_2^*(\alpha_j)"] \arrow[d, "\varphi_{ij}"'] & \mathrm{pr}_2^* \eta_j \arrow[d, "\psi_{ij}"] \\
\mathrm{pr}_1^* \xi_i \arrow[r, "\mathrm{pr}_1^*(\alpha_i)"'] & \mathrm{pr}_1^* \eta_i.
\end{tikzcd}$$

We denote by $\mathscr{F}(\mathcal{U}/U)$ the category of objects with descent data for the covering $\mathcal{U}$.

Example 5.14 (From an object to a descent data)
Let ξ be an object of $\mathscr{F}(U)$. We can construct an object with a descent data for the covering $\mathcal{U} = \left\{ U_i \xrightarrow{\varphi_i} U \right\}$ as follows:

- We let $\xi_i = \varphi_i^*(\xi)$.
- Since $\varphi_i \circ \mathrm{pr}_1 = \varphi_j \circ \mathrm{pr}_2$, we get an isomorphism $\varphi_{ij} : \mathrm{pr}_2^* \xi_j \longrightarrow \mathrm{pr}_1^* \xi_1$ via $\alpha_{\mathrm{pr}_1, \varphi_i}$ and $\alpha_{\mathrm{pr}_2, \varphi_j}$.

Now, if $\varphi : \xi \longrightarrow \eta$ is a morphism in $\mathscr{F}(U)$, we get a morphism $\{\alpha_i : \xi_i \longrightarrow \eta_i\}$ between $\left(\{\xi_i\}, \{\varphi_{ij}\}\right)$ and $\left(\{\eta_i\}, \{\psi_{ij}\}\right)$ by letting $\alpha_i = \varphi_i^*(\varphi)$. Since each φ_i^* is a functor, we have a functor from $\mathscr{F}(U)$ to $\mathscr{F}(\mathcal{U}/U)$.

Definition 5.15 (Effective descent data)
A data descent $\{\xi_i\}$ is called effective *if there exists some object ξ in $\mathscr{F}(U)$ which induces (up to isomorphism) the family $\{\xi_i\}$ (via the functor defined in the previous example).*

Theorem 5.16
Let U be a scheme and let $\left\{ U_i \longrightarrow U \right\}$ be an fpqc covering of U. The functor $\mathfrak{Coh}(U) \longrightarrow \mathfrak{Coh}(\mathcal{U}/U)$ (see Example 5.10 for a definition of $\mathfrak{Coh}$) is an equivalence of categories. In particular, every data descent of $\mathcal{O}_{U_i}$-module quasi-coherent is effective.

Proof. See [BLR90, Theorem 6.4]. □

Theorem 5.17
Let S be a scheme and let X be a S-scheme. Then, the functor of points $h_X = \mathrm{Hom}_S(-, X)$ is a sheaf for the fpqc topology.

Proof. See [Bro11, Theorem 2.2.5] □

6 Fppf topology and representability

6.1 The fppf site

Definition 6.1 (Fppf covering)
Let X be a scheme. An fppf covering *of X is given by a family of morphisms $f_i : X_i \longrightarrow X$ such that each f_i is flat and locally of finite presentation and such that $\bigcup_i \operatorname{im} f_i = X$.*

Proposition 6.2
The collection of fppf coverings of a scheme S satisfies the conditions of a site.

Proof. We already know that the coverings by flat morphisms satisfy these properties. Moreover, we know that composition of morphisms of locally of finite presentation is again locally of finite presentation and that this property is stable under base change (see, for example, [Aut, Morphisms of schemes, 24.19]). □

Definition 6.3 (The fppf site)
Let S be a scheme. The fppf site *on S is the category $\mathfrak{Sch}(S)$ together with the fppf coverings.*

6.2 Fppf topology and representability

In this section, S is a scheme and X is a S-scheme.

Definition 6.4 (Picard group of a ringed space)
Let $(X, \mathcal{O}_X)$ be a scheme. The set of isomorphism classes of invertible $\mathcal{O}_X$-modules can be endowed with a structure of abelian group whose law group is the tensor product over $\mathcal{O}_X$. The details can be found in [Har77, II.6].

Definition 6.5 (Relative Picard functor)
We have a contravariant functor

$$\begin{aligned} P_{X/S} : \mathfrak{Sch}(S) &\longrightarrow \mathbf{Set} \\ T &\longmapsto \operatorname{Pic}(X \times_S T). \end{aligned}$$

A morphism of S-schemes $f : T \longrightarrow T'$ gives rise to a morphism of schemes $\tilde{f} : X \times_S T \longrightarrow X \times_S T'$ and then to a map from $\operatorname{Pic}(X \times_S T')$ to $\operatorname{Pic}(X \times_S T)$: an $\mathcal{O}_T$-module $\mathscr{F}$ is sent to $\tilde{f}^*(\mathscr{F})$, where $\tilde{f}^*$ is the usual pullback (see Example 5.10).

Remark 6.6
In fact, the morphism $\tilde{f}^* : \operatorname{Pic}(X \times_S T') \longrightarrow \operatorname{Pic}(X \times_S T)$ is a group homomorphism, but we forget the group structure here.

Theorem 6.7
Let F be a contravariant representable functor on $\mathfrak{Sch}(S)$ with values in **Set**. *Then, F is a sheaf with respect to the fpqc topology (and thus for the fppf, étale and Zariski topology).*

Proof. See [BLR90, Proposition 8.1]. □

Since the relative Picard functor may failed to be a sheaf (even for the Zariski topology) it may not be representable. However, there are some nice situations where the sheafification of $P_{X/S}$ (see Theorem 3.27) is representable.

Definition 6.8
The sheafification of the relative Picard functor (see Theorem 3.27) is denoted by $\operatorname{Pic}_{X/S}$. For any S-scheme T, we call $\operatorname{Pic}_{X/S}(T)$ the relative Picard group *of $X \times_S T$ over T.*

In order to state the theorem, we recall some notations and definitions.

Notation 6.9
For any ring R, we denote by $\mathbb{P}^n_R$ the scheme $\operatorname{Proj} R[x_0, \ldots, n]$.

Definition 6.10 (Projective space over a scheme)
Let X be a scheme and let $n \in \mathbb{N}$. The projective n-space over X *is the scheme* $\mathbb{P}^n_X := \mathbb{P}^n_{\mathbb{Z}} \times_{(\operatorname{Spec} \mathbb{Z})} X$.

Definition 6.11 (Projective morphism)
A morphism of schemes $f : X \longrightarrow Y$ is projective *if there exists $n \in \mathbb{N}$ and a closed immersion i such that the following diagram is commutative*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & \mathbb{P}^n_Y \arrow[ur] & \end{tikzcd}$$

where $\mathbb{P}^n_Y \longrightarrow Y$ is the canonical morphism.

Theorem 6.12
Let X and S be two locally noetherian schemes. Let $f : X \longrightarrow S$ be a projective morphism of finite presentation. Moreover, suppose that f is flat and that all geometric fibers are integral. Then, the sheaf $\operatorname{Pic}_{X/S}$ is representable by a separated S-scheme which is locally of finite type over S.

Proof. See [Gro61, Theorem 3.1]. □

A Some results of algebra

Proposition A.1
Let R be an artinian ring. Then:

(*i*) *Every prime ideal is maximal.*

(*ii*) $\operatorname{Spec} R$ *is finite.*

Proof. (i) Let P be a prime ideal of R and r be any element such that $r \notin P$. Consider the sequence

$$P + rR \supset P + r^2R \supset \ldots \supset P + r^nR \supset \ldots.$$

Since R is artinian, there exists some $n \in \mathbb{N}$ such that $P + r^nR = P + r^{n+1}R$. Hence, there exists elements $s \in R$ and $p \in P$ such that $1 - rs = p$, that is $1 = p + rs \in P + rR$. Therefore, P is maximal.

(ii) The previous point implies that it is sufficient to show that the set of maximal ideals of R is finite.
Let $M_1, \ldots, M_{n+1}$ be different maximal ideals of R. We want to show that

$$M_1 \cdot \ldots \cdot M_{n+1} \subsetneq M_1 \cdot \ldots \cdot M_n$$

Since the M_i are maximal ideals, we can find for every $i = 1, \ldots, n$ an element $f_i \in M_i \setminus M_{n+1}$. If we have equality in the previous inclusion, then we have

$$m_1 \cdot \ldots \cdot m_m \in M_1 \cdot \ldots \cdot M_n = M_1 \cdot \ldots \cdot M_{n+1} \subset M_{n+1},$$

which is impossible.
Now, suppose that the set of maximal ideals of R is infinite. Then, we can form an infinite strictly decreasing sequence of ideals

$$M_1 \supsetneq M_1M_2 \supsetneq M_1M_2M_3 \supsetneq \ldots,$$

which contradicts the fact that R is artinian.

□

Proposition A.2
Let $\varphi : A \longrightarrow B$ be a morphism of rings and I be an ideal of A. We denote by I^e the ideal generated by $\varphi(I)$. Then, we have $B/_{I^e} \cong B \otimes_A A/_I$.

Proof. We construct the following morphisms:

$$\begin{tikzcd}
B \arrow[r, "\psi"] \arrow[d, "\pi"'] & B \otimes_A A/_I \arrow[dl, shift left, "\overline{\eta}"] \\
B/_{I^e} \arrow[ur, shift left, "\overline{\psi}"] & B \times A/_I . \arrow[l, "\eta"] \arrow[u, "\tau"']
\end{tikzcd}$$

First, we define $\psi(b) = b \otimes \overline{1}$ which induces $\overline{\psi}$. Then, we set $\eta\big(b, \overline{a}\big) = \overline{a \cdot b}$ and this morphism induces $\overline{\eta}$. Finally, one can check that $\overline{\psi}$ and $\overline{\eta}$ are inverses to each other. □

Lemma A.3
Let M be an R-module. Then, $M = 0$ if and only if $M_{\mathfrak{m}} = 0$ for each $\mathfrak{m} \in \operatorname{MaxSpec} R$.

Proof. If $M = 0$, then $M_{\mathfrak{m}} \cong M \otimes R_{\mathfrak{m}} = 0$. Now, suppose that $M_{\mathfrak{m}} = 0$ for each $\mathfrak{m} \in \operatorname{MaxSpec} R$ and suppose that $M \neq 0$. Choose $m \in M \setminus \{0\}$ and consider $\operatorname{ann}(m) = \{r \in R : rm = 0\}$. This ideal is a proper ideal of R, since 1 does not belong to it. Therefore, it is contained in some maximal ideal $\mathfrak{m}$. Since $\frac{m}{1} = 0$, there exists $r \in R \setminus M$ such that $r = 0$. Contradiction. □

Lemma A.4
Let B be a ring and write $\operatorname{Spec} B = \bigcup_i D(b_i)$, where $D(b_i)$ is the principal open subset $B \setminus \mathcal{V}(\langle b_i \rangle)$. If each B_{b_i} is a A-algebra of finite type, then so is B.

Proof. First, remark that since $\operatorname{Spec} B$ is compact, then we can suppose that the number of the $D(b_i)$ is finite. By hypothesis, there exists for each i a set of elements $\frac{b_{ij}}{b_i^{k_i}}$ of B_{b_i} which generate B_{b_i} as an A-algebra. Now, we let C denote the A-algebra generated by the $b_{ij}, b_i, b_i^{k_i}$, that is: C is a finitely generated A-algebra, it contains the b_i and $C_{b_i} \supset B_{b_i}$ for each i. Since the $D(b_i)$ form a covering of $\operatorname{Spec} B$, there exists some b_i' such that $1 = \sum_i b_i b_i'$. We let D the sub-A-algebra of B which contains both C and the b_i'. For each natural number k, taking the k-th power of the expression $1 = \sum_i b_i b_i'$ gives a relation $1 = \sum_i b_i^k d_{i,k}$ for some $d_{i,k} \in D$. Now, we want to show that $B = D$. Let $b \in B$. For each i, we have $\frac{b}{1} \in B_{b_i} \subset C_{b_i}$. Hence, there exists some $c_i \in C$ and some $m_i \in \mathbb{N}$ such that $b_i^{m_i} b = b_i^{m_i} c_i$ for every i. Taking $m = \max m_i$ gives us $b = \sum_i (b b_i^m) d_{i,k} \in D$, as required. □

Proposition A.5
Let R be a local ring with and let I be an ideal of R. Let M be a finitely generated R-module. Suppose that $\overline{m}_1, \ldots, \overline{m}_t \in M/_{IM}$ generate $M/_{\mathfrak{m}M}$ as a $R/_I$-module. Then, $m_1, \ldots, m_r$ generate M as a R-module.

Proof. Let N be the submodule of M generated by $m_1, \ldots, m_t$. Since the composition $N \longrightarrow M \longrightarrow M/_{\mathfrak{m}M}$ is surjective, we have $M = N + \mathfrak{m}M$. Hence, Nakayama's lemma implies that $N = M$, as required. □

Corollary A.6
Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let M be a finitely generated R-module. Then, $M \otimes_R k \cong 0$ implies $M = 0$.

Corollary A.7
Let R be a local ring and let I be an ideal of R. Let M be a finitely generated R-module. Then, $I/_{I^2} \cong 0$ implies $I = 0$.

Table of notations

$\mathfrak{Coh}(X)$	The category of quasi-coherents modules over the scheme X
Grp	The category of groups
$\operatorname{ht}\mathfrak{p}$	Height of the prime ideal $\mathfrak{p}$
$\kappa(x)$	Residue field at x
$\mathfrak{m}_x$	Maximal ideal of $\mathcal{O}_{X,x}$
$\operatorname{Nat}(F,G)$	Natural transformations between the functors F and G
$\operatorname{Pic}(X)$	Picard group of X
${}_R\mathbf{Mod}$	The category of left R-modules
Rng	The category of rings
$\mathfrak{Sch}$	Category of schemes
$\mathfrak{Sch}_{\text{ét}}$	Category of schemes with étale morphisms
$\mathfrak{Sch}_{\text{oi}}$	Category of schemes with open immersions as morphisms
$\mathfrak{Sch}(S)$	Category of schemes over the scheme S
Set	The category of sets

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